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Multiloop Amplitudes and Vanishing Theorems using the Pure Spinor Formalism for the Superstring

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A ten-dimensional super-Poincaré covariant formalism for the superstring was recently developed which involves a BRST operator constructed from superspace matter variables and a pure spinor ghost variable. A super-Poincaré covariant prescription was defined for computing tree amplitudes and was shown to coincide with the standard RNS prescription.

In this paper, picture-changing operators are used to define functional integration over the pure spinor ghosts and and to construct a suitable b ghost. A super-Poincaré covariant prescription is then given for the computation of N-point multiloop amplitudes. One can easily prove that massless N-point multiloop amplitudes vanish for $N < 4$, confirming the perturbative finiteness of superstring theory. One can also prove the Type IIB S-duality conjecture that R^4 terms in the effective action receive no perturbative contributions above one loop.

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1. Introduction

The computation of multiloop amplitudes in superstring theory has many important applications such as verifying perturbative finiteness and testing duality conjectures. Nevertheless, this subject has received little attention over the last fifteen years, mainly because of difficulties in computing multiloop amplitudes using either the Ramond-Neveu-Schwarz (RNS) or Green-Schwarz (GS) formalism.

In the RNS formalism, spacetime supersymmetric amplitudes are obtained after summing over spin structures, which can be done explicitly only when the number of loops and external states is small [1]. Since there are divergences near the boundary of moduli space before summing over spin structures, surface terms in the amplitude expressions need to be treated with care [2][3] [4] [5]. Furthermore, the complicated nature of the Ramond vertex operator in the RNS formalism [6] makes it difficult to compute amplitudes involving external fermions or Ramond-Ramond bosons. For these reasons, up to now, explicit multiloop computations in the RNS formalism have been limited to four-point two-loop amplitudes involving external Neveu-Schwarz bosons [7][5].[2]

In the GS formalism, spacetime supersymmetry is manifest but one needs to fix light-cone gauge and introduce non-covariant operators at the interaction points of the Mandelstam string diagram[9][10][11]. Because of complications caused by these non-covariant interaction point operators [12], explicit amplitude expressions have been computed using the light-cone GS formalism only for four-point tree and one-loop amplitudes [9].[3]

Four years ago, a new formalism for the superstring was proposed [14][15] with manifest ten-dimensional super-Poincaré covariance. In conformal gauge, the worldsheet action is quadratic and physical states are defined using a BRST operator constructed from superspace matter variables and a pure spinor ghost variable. A super-Poincaré covariant prescription was given for computing N-point tree amplitudes, which was later shown to coincide with the standard RNS prescription [16][17]. It was also proven that the BRST cohomology reproduces the correct superstring spectrum [18] and that BRST invariance in

[2] Danilov [8] has claimed to be able to compute RNS amplitudes for arbitrary genus, however, this author has been unable to understand his methods.

[3] Although multiloop GS expressions were obtained by Restuccia and Taylor in [13], this author does not think that they correctly took into account the contact terms between interaction-point operators. Note that the N-point tree amplitudes proposed by Mandelstam in [11] were derived using unitarity arguments and were not directly computed from the GS formalism.

a curved supergravity background implies the low-energy superspace equations of motion for the background superfields [19][20].

Because of the pure spinor constraint satisfied by the worldsheet ghosts, it was not known how to define functional integration in this formalism. For this reason, the tree amplitude prescription in [14] relied on BRST cohomology for defining the correct normalization of the worldsheet zero modes. Furthermore, there was no natural b ghost in this formalism, which made it difficult to define amplitudes in a worldsheet reparameterization-invariant manner. Because of these complications, it was not clear how to compute loop amplitudes using this formalism and other groups looked for ways of relaxing the pure spinor constraint without modifying the BRST cohomology [21] [22][23][24].

In this paper, it will be shown how to perform functional integration over the pure spinor ghosts[4] by defining a Lorentz-invariant measure and introducing appropriate "picture-changing" operators.[5] These picture-changing operators will then be used to construct a substitute for the b ghost in a non-zero picture. With these ingredients, it is straightforward to generalize the tree amplitude prescription of [14] to a super-Poincaré covariant prescription for N-point g-loop amplitudes. So there is no need to relax the pure spinor constraint for the covariant computation of superstring amplitudes.

The need for picture-changing operators[6] in this formalism is not surprising since, like the bosonic (β, γ) ghosts in the RNS formalism [6], the pure spinor ghosts are chiral

[4] Some features of this functional integration method will appear in a separate paper with Sergei Cherkov [25].

[5] Using the pure spinor version of the superparticle, Chesterman has recently considered superparticle states with non-standard boundary conditions for the pure spinor ghosts [23][24]. In independent work which appeared last month [24], Chesterman showed that these states are related to standard superparticle states by an operator ψ_{-11} which plays the role of the picture-lowering operator described in this paper.

Also, in independent work [26] which was announced after a seminar on this paper, Grassi, Policastro and van Nieuwenhuizen used functional integration to define the measure factor in their quantization approach without pure spinors. It would be interesting to relate their functional integration method with the method described here.

[6] One can also use picture-changing operators to construct a cubic pure spinor version of open superstring field theory. However, as in the cubic RNS version of open superstring field theory [27], the action is expected to have gauge-invariance anomalies due to picture-changing operators at the string midpoint [28]. It should be stressed that these anomalies in cubic superstring field theory are caused by the use of picture-changing operators in the presence of off-shell states and do not imply surface-term ambiguities in on-shell multiloop amplitudes.

bosons with worldsheet zero modes. For g-loop amplitudes, the use of standard "picture-zero" vertex operators implies that one needs to insert 11 "picture-lowering" operators and $11g$ "picture-raising" operators to absorb the zero modes of the 11 pure spinor ghosts. As in the RNS formalism, the worldsheet derivatives of these picture-changing operators are BRST trivial so, up to possible surface terms, the amplitudes are independent of their locations on the worldsheet. But unlike the RNS formalism, there is no need to sum over spin structures, so there are no divergences at the boundary of moduli space and surface terms can be safely ignored in the loop amplitude computations.

Although the explicit computation of arbitrary loop amplitudes is complicated, one can easily prove certain vanishing theorems by counting zero modes of the fermionic superspace variables. For example, S-duality of the Type IIB superstring implies that R^4 terms in the low-energy effective action receive no perturbative corections above one-loop [29]. After much effort, this was recently verified in the RNS formalism at two-loops [7][5]. Using the formalism described here, this S-duality conjecture can be easily verified for all loops.

Similarly, one can easily prove the non-renormalization theorem that massless N-point multiloop amplitudes vanish whenever $N < 4$. Assuming factorization, this non-renormalization theorem implies the absence of divergences near the boundary of moduli space [4][30]. The boundary of moduli space includes two types of degenerate surfaces: surfaces where the radius R of a handle shrinks to zero, and surfaces which split into two worldsheets connected by a thin tube. As explained in [4], the first type of degenerate surface does not lead to divergent amplitudes since, after including the $\log(R)$ dependence coming from integration over the loop momenta, the amplitude integrand diverges slower than $1/R$. The second type of degenerate surface can lead to a divergent amplitude if there is an onshell state propagating along the thin tube between the two worldsheets. But when all external states are on one of the two worldsheets, vanishing of the one-point function implies the absence of this divergence. And when all but one of the external states are on one of the two worldsheets, vanishing of the two-point function implies the absence of this divergence. Finally, when there are at least two external states on each of the two worldsheets, the divergence can be removed by analytic continuation of the external momenta [4]. Note that vanishing of the three-point function is not required for finiteness.

So if there are no unphysical divergences in the interior of moduli space[7], this non-renormalization theorem implies that superstring multiloop amplitudes are perturbatively finite. Previous attempts to prove this non-renormalization theorem using the RNS formalism [32] were unsuccessful because they ignored unphysical poles of the spacetime supersymmetry currents [2] and incorrectly assumed that the integrand of the scattering amplitude was spacetime supersymmetric. Using the GS formalism, there are arguments for the non-renormalization theorem [33], however, these arguments do not rule out the possibility of unphysical divergences in the interior of moduli space from contact term singularities between light-cone interaction point operators [12]. Mandelstam was able to overcome this obstacle and prove finiteness [31] by combining different features of the RNS and GS formalisms. However, the finiteness proof here is more direct than the proof of [31] since it is derived from a single formalism.

In section 2 of this paper, the super-Poincaré invariant pure spinor formalism of [14] is reviewed. The first subsection reviews the worldsheet action for the Green-Schwarz-Siegel matter variables and the OPE's for the pure spinor ghosts. The second subsection reviews the BRST operator and shows how physical states are described by the BRST cohomology. The third subsection reviews the computation of tree amplitudes using a measure factor determined by cohomology arguments.

In section 3, functional integration over the pure spinor ghosts is defined with the help of picture-changing operators. The first subsection shows how to define Lorentz-invariant measure factors for integration over the pure spinor ghosts and their conjugate momenta. The second subsection introduces picture-raising and picture-lowering operators which are necessary for functional integration over the bosonic ghosts. The third subsection shows that by inserting picture-lowering operators, the tree amplitudes of section 2 can be computed using standard functional integration techniques.

In section 4, a composite b ghost is defined by requiring that the BRST variation of the b ghost is a picture-raised version of the stress tensor. The first subsection introduces a chain of operators of $+2$ conformal weight which are useful for explicitly constructing the b ghost. The second subsection shows how the various terms in the composite b ghost can be expressed in terms of these operators.

[7] In light-cone gauge, unphysical divergences in the interior of moduli space could come from singularities between colliding interaction points [12][31]. In conformal gauge, there are no obvious potential sources for these unphysical divergences in the interior of moduli space since the amplitudes are independent (up to surface terms) of the locations of picture-changing operators.

In section 5, a super-Poincaré covariant prescription is given for N-point g-loop amplitudes. The partition function for the matter and ghost variables precisely cancel in this prescription, so one only needs to compute correlation functions. The first and second subsections show how to compute correlation functions for the matter and ghost variables by separating off the zero modes and using the free-field OPE's to functionally integrate over the non-zero modes. The third subsection shows how to integrate over the zero modes using the measure factor defined in section 3 for the pure spinor ghosts and their conjugate momenta.

Finally, in section 6, the four-point one-loop amplitude is computed and certain vanishing theorems are proven using the multiloop prescription. In the first subsection, the structure of Type II massless vertex operators is reviewed. In the second subsection, the non-renormalization theorem for less than four massless states is proven by zero-mode counting. In the third subsection, the four-point massless one-loop amplitude is explicitly computed up to an overall constant. And in the fourth subsection, it is proven by zero-mode counting that the R^4 term in the low-energy effective action does not receive perturbative corrections above one loop.

2. Review of Super-Poincaré Covariant Pure Spinor Formalism

2.1. Worldsheet action

The worldsheet variables in the Type IIB version of this formalism include the Green-Schwarz-Siegel [34][35] matter variables $(x^m, \theta^\alpha, p_\alpha; \overline{\theta}^\alpha, \overline{p}_\alpha)$ for $m = 0$ to 9 and $\alpha = 1$ to 16, and the pure spinor ghost variables $(\lambda^\alpha, w_\alpha; \overline{\lambda}^\alpha, \overline{w}_\alpha)$ where λ^α and $\overline{\lambda}^\alpha$ are constrained to satisfy the pure spinor conditions

$$\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta = 0, \quad \overline{\lambda}^\alpha(\gamma^m)_{\alpha\beta}\overline{\lambda}^\beta = 0 \tag{2.1}$$

for $m = 0$ to 9. $(\gamma^m)_{\alpha\beta}$ and $(\gamma^m)^{\alpha\beta}$ are 16×16 symmetric matrices which are the off-diagonal blocks of the 32×32 ten-dimensional Γ-matrices and satisfy $(\gamma^{(m})_{\alpha\beta}(\gamma^{n)})^{\beta\gamma} = 2\eta^{mn}\delta_\alpha^\gamma$. For the Type IIA version of the formalism, the chirality of the spinor indices on the right-moving variables is reversed, and for the heterotic version, the right-moving variables are the same as in the RNS formalism.

In conformal gauge, the worldsheet action is

$$S = \int d^2z[-\frac{1}{2}\partial x^m\overline{\partial}x_m - p_\alpha\overline{\partial}\theta^\alpha - \overline{p}_\alpha\partial\overline{\theta}^\alpha + w_\alpha\overline{\partial}\lambda^\alpha + \overline{w}_\alpha\partial\overline{\lambda}^\alpha] \tag{2.2}$$

where λ^α and $\overline{\lambda}^\alpha$ satisfy (2.1). The OPE's for the matter variables are easily computed to be

$$x^m(y)x^n(z) \to -\eta^{mn}\log|y-z|^2, \quad p_\alpha(y)\theta^\beta(z) \to (y-z)^{-1}\delta_\alpha^\beta, \tag{2.3}$$

however, the pure spinor constraint on λ^α prevents a direct computation of its OPE's with w_α. As discussed in [14], one can solve the pure spinor constraint and express λ^α in terms of eleven unconstrained free fields which manifestly preserve a U(5) subgroup of the (Wick-rotated) Lorentz group. Although the OPE's of the unconstrained variables are not manifestly Lorentz-covariant, all OPE computations involving λ^α can be expressed in a manifestly Lorentz-covariant manner. So the non-covariant unconstrained description of pure spinors is useful only for verifying certain coefficients in the Lorentz-covariant OPE's.

Because of the pure spinor constraint on λ^α, the worldsheet variables w_α contain the gauge invariance

$$\delta w_\alpha = \Lambda^m(\gamma_m\lambda)_\alpha, \tag{2.4}$$

so 5 of the 16 components of w_α can be gauged away. To preserve this gauge invariance, w_α can only appear in the gauge-invariant combinations

$$N_{mn} = \frac{1}{2}w_\alpha(\gamma_{mn})^\alpha{}_\beta\lambda^\beta, \quad J = w_\alpha\lambda^\alpha, \tag{2.5}$$

which are the Lorentz currents and ghost current. As shown in [17] and [18] using either the U(5) or SO(8) unconstrained descriptions of pure spinors[8], N_{mn} and J satisfy the Lorentz-covariant OPE's

$$N_{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}(y-z)^{-1}(\gamma_{mn}\lambda)^\alpha, \quad J(y)\lambda^\alpha(z) \to (y-z)^{-1}\lambda^\alpha, \tag{2.6}$$

$$N^{kl}(y)N^{mn}(z) \to -3(y-z)^{-2}(\eta^{n[k}\eta^{l]m}) + (y-z)^{-1}(\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}),$$

$$J(y)J(z) \to -4(y-z)^{-2}, \quad J(y)N^{mn}(z) \to \text{regular},$$

$$N_{mn}(y)T(z) \to (y-z)^{-2}N_{mn}(z), \quad J(y)T(z) \to -8(y-z)^{-3} + (y-z)^{-2}J(z),$$

where

$$T = -\frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha \tag{2.7}$$

[8] In reference [18] for the SO(8) description, the ghost current J was not discussed. In terms of the SO(8)-covariant variables of [18], $J = -2bc + s^a r^a - 2\sum_{n=0}^\infty [n v^j_{(n)} w^j_{(n)} + (n+\frac{1}{2})t^{\dot a}_{(n)}u^{\dot a}_{(n)}]$. Using the summation method described in [18], it is straightforward to check that J satisfies the OPE's described here.

is the left-moving stress tensor. From the OPE's of (2.6), one sees that the pure spinor condition implies that the levels for the Lorentz and ghost currents are -3 and -4, and that the ghost-number anomaly is -8. Note that the total Lorentz current $M^{mn} = -\frac{1}{2}(p\gamma^{mn}\theta) + N^{mn}$ has level $k = 4 - 3 = 1$, which coincides with the level of the RNS Lorentz current $M^{mn} = \psi^m\psi^n$. The ghost-number anomaly of -8 will be related in section 3 to the pure spinor measure factor.

The stress tensor of (2.7) has no central charge since the $(+10-32)$ contribution from the $(x^m, \theta^\alpha, p_\alpha)$ variables is cancelled by the $+22$ contribution from the eleven independent $(\lambda^\alpha, w_\alpha)$ variables. From its OPE's with N_{mn} and J, one learns that the stress tensor can be expressed in Sugawara form as[9]

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + \frac{1}{10} : N^{mn} N_{mn} : -\frac{1}{8} : JJ : + \partial J \tag{2.8}$$

where the level -3 SO(9,1) current algebra contributes -27 to the central charge and the ghost current J contributes $+49$.

2.2. Physical states

Physical open string states in this formalism are defined as super-Poincaré covariant states of ghost-number $+1$ in the cohomology of the nilpotent BRST-like operator

$$Q = \oint \lambda^\alpha d_\alpha \tag{2.9}$$

where

$$d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\ \gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta \tag{2.10}$$

is the supersymmetric Green-Schwarz constraint. As shown by Siegel [35], d_α satisfies the OPE's

$$d_\alpha(y) d_\beta(z) \to -(y-z)^{-1}\gamma^m_{\alpha\beta}\Pi_m, \quad d_\alpha(y)\Pi^m(z) \to (y-z)^{-1}\gamma^m_{\alpha\beta}\partial\theta^\beta(z), \tag{2.11}$$
$$d_\alpha(y)\partial\theta^\beta(z) \to (y-z)^{-2}\delta^\beta_\alpha, \quad \Pi^m(y)\Pi^n(z) \to -(y-z)^{-2}\eta^{mn},$$

where $\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$ is the supersymmetric momentum and

$$q_\alpha = \oint (p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\ \gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta) \tag{2.12}$$

[9] There is a typo in the sign of the ∂J term in references [17] and [15].

is the supersymmetric generator satisfying

$$\{q_\alpha, q_\beta\} = \gamma^m_{\alpha\beta} \oint \partial x_m, \quad [q_\alpha, \Pi^m(z)] = 0, \quad \{q_\alpha, d_\beta(z)\} = 0. \tag{2.13}$$

To compute the massless spectrum of the open superstring[10], note that the most general vertex operator with zero conformal weight at zero momentum and +1 ghost-number is

$$V = \lambda^\alpha A_\alpha(x, \theta), \tag{2.14}$$

where $A_\alpha(x,\theta)$ is a spinor superfield depending only on the worldsheet zero modes of x^m and θ^α. Using the OPE that $d_\alpha(y)\ f(x(z), \theta(z)) \to (y-z)^{-1} D_\alpha f$ where

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2}\theta^\beta \gamma^m_{\alpha\beta}\partial_m \tag{2.15}$$

is the supersymmetric derivative, one can easily check that $QV = 0$ and $\delta V = Q\Lambda$ implies that $A_\alpha(x,\theta)$ must satisfy $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$ with the gauge invariance $\delta A_\alpha = D_\alpha \Lambda$. But $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$ implies that

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m \tag{2.16}$$

for some vector superfield A_m with the gauge transformations

$$\delta A_\alpha = D_\alpha \Lambda, \quad \delta A_m = \partial_m \Lambda. \tag{2.17}$$

In components, one can use (2.16) and (2.17) to gauge A_α and A_m to the form

$$A_\alpha(x,\theta) = e^{ik\cdot x}(\frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha + ...), \tag{2.18}$$

$$A_m(x,\theta) = e^{ik\cdot x}(a_m + (\xi\gamma^m\theta) + ...),$$

where $k^2 = k^m a_m = k^m(\gamma_m\xi)_\alpha = 0$, and ... involves products of k_m with a_m or ξ^α. So (2.16) and (2.17) are the equations of motion and gauge invariances of the ten-dimensional super-Maxwell multiplet, and the cohomology at ghost-number +1 of Q correctly describes the massless spectrum of the open superstring [36].

To compute the massive spectrum, one needs to consider the cohomology of vertex operators which have non-zero conformal weight at zero momentum. This was done with Chandia in [37] for the massive spin-two multiplet and gave for the first time its equations

[10] Massless vertex operators for the closed superstring will be reviewed in subsection (6.1).

of motion in ten-dimensional superspace. To prove that the cohomology of Q reproduces the superstring spectrum at arbitrary mass level, the SO(8)-covariant description was used in [18] to solve the pure spinor constraint, and the resulting BRST cohomology was shown to be equivalent to the light-cone GS spectrum.

In addition to describing the spacetime fields at ghost-number +1, the cohomology of Q can also be used to describe the spacetime ghosts at ghost-number zero, the spacetime antifields at ghost-number +2, and the spacetime antighosts at ghost-number +3 [38][39]. For example, the super-Yang-Mills ghost at ghost-number zero is described by the vertex operator $V = \Lambda$, the super-Yang-Mills antifields at ghost-number two are described by the vertex operator $V = \lambda^\alpha \lambda^\beta A^*_{\alpha\beta}(x,\theta)$, and the Yang-Mills antighost at ghost-number three is described by the vertex operator $V = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. As was shown in [39], the conditions $QV = 0$ and $\delta V = Q\Lambda$ imply the correct equations of motion and gauge invariances for these ghosts, antifields and antighosts.

2.3. Tree-level prescription

As in the bosonic string, the prescription for N-point open string tree amplitudes in this formalism requires three dimension-zero vertex operators V and $N-3$ dimension-one vertex operators U which are integrated over the real line. Normally, one defines the dimension-one vertex operators by $U(z) = \{\oint b, V(z)\}$ where $b(z)$ is the dimension-two field satisfying $\{Q, b(z)\} = T(z)$. Since $QV = 0$ and $[\oint T, V(z)] = \partial V(z)$, this relation implies that $QU = \partial V$.

In this formalism of the superstring, there are no states of negative ghost number since the variable w_α can only appear through the ghost-number zero operators N_{mn} and J. So one cannot construct a b ghost satisfying $\{Q, b\} = T$. Nevertheless, since there is no BRST cohomology for unintegrated dimension-one operators, one is guaranteed that $QV = 0$ implies that ∂V can be written as QU for some U. For example, for the super-Maxwell vertex operator $V = \lambda^\alpha A_\alpha$, one can check that

$$U = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + \frac{1}{2} N^{mn} \mathcal{F}_{mn}(x,\theta) \tag{2.19}$$

satisfies $QU = \partial(\lambda^\alpha A_\alpha)$ where $A_m = \frac{1}{8} D_\alpha \gamma_m^{\alpha\beta} A_\beta$ is the vector gauge superfield, $W^\beta = \frac{1}{10}\gamma_m^{\alpha\beta}(D_\alpha A^m - \partial^m A_\alpha)$ is the spinor superfield strength, and $\mathcal{F}_{mn} = \frac{1}{8} D_\alpha (\gamma_{mn})^\alpha{}_\beta W^\beta = \partial_{[m} A_{n]}$ is the vector superfield strength.

In reference [14], open string tree amplitudes were defined by the correlation function

$$\mathcal{A} = \langle\ V_1(z_1)\ V_2(z_2)\ V_3(z_3)\ \int dz_4 U_4(z_4) ... \int dz_N U_N(z_N)\ \rangle. \tag{2.20}$$

To compute this correlation function, the OPE's of (2.6) and (2.11) were used to perform the functional integration over the non-zero modes of the worldsheet variables. Since N_{mn}, J and d_α are fields of $+1$ conformal weight with no zero modes on a sphere, the dependence of the correlation function on their locations is completely determined by the singularities in their OPE's. For example, the OPE's of $d_\alpha(z)$ imply that[11]

$$\langle d_\alpha(z)\Pi^m(u)\partial\theta^\beta(v)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle = \tag{2.21}$$

$$\gamma^m_{\alpha\rho}(z-u)^{-1}\langle\partial\theta^\rho(u)\partial\theta^\beta(v)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle$$

$$+\delta^\beta_\alpha(z-v)^{-2}\langle\Pi^m(u)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle$$

$$+\gamma^n_{\alpha\gamma}(z-w)^{-1}\langle\Pi^m(u)\partial\theta^\beta(v)\Pi^n(w)A_\delta(x(y),\theta(y))\rangle$$

$$+(z-y)^{-1}\langle\Pi^m(u)\partial\theta^\beta(v)d_\gamma(w)D_\alpha A_\delta(x(y),\theta(y))\rangle.$$

And the OPE's of $N_{mn}(z)$ imply that

$$\langle N_{mn}(z)N_{pq}(u)\lambda^\alpha(v)\lambda^\beta(w)\lambda^\gamma(y)\rangle = \tag{2.22}$$

$$= (z-u)^{-1}\langle(\eta_{p[n}N_{m]q}(u) - \eta_{q[n}N_{m]p}(u))\lambda^\alpha(v)\lambda^\beta(w)\lambda^\gamma(y)\rangle$$

$$-3(z-u)^{-2}\eta_{q[m}\eta_{n]p}\langle\lambda^\alpha(v)\lambda^\beta(w)\lambda^\gamma(y)\rangle$$

$$+\frac{1}{2}(z-v)^{-1}\langle N_{pq}(u)(\gamma_{mn}\lambda(v))^\alpha\lambda^\beta(w)\lambda^\gamma(y)\rangle$$

$$+\frac{1}{2}(z-w)^{-1}\langle N_{pq}(u)\lambda^\alpha(v)(\gamma_{mn}\lambda(w))^\beta\lambda^\gamma(y)\rangle$$

$$+\frac{1}{2}(z-y)^{-1}\langle N_{pq}(u)\lambda^\alpha(v)\lambda^\beta(w)(\gamma_{mn}\lambda(y))^\gamma\rangle.$$

After using their OPE's to remove all N_{mn}'s, J's and d_α's from the correlation function, one can replace all remaining λ^α and θ^α variables by their zero modes. But since it was not known how to perform the functional integration over the remaining zero modes of

[11] To keep supersymmetry manifest, it is convenient to use the OPE's of (2.11) for d_α instead of using the free-field OPE's of (2.3) for p_α.

the worldsheet scalars λ^α and θ^α, an ansatz had to be used for deciding which zero modes of λ^α and θ^α need to be present for non-vanishing amplitudes.

For tree amplitudes in bosonic string theory, the zero-mode prescription coming from functional integration is

$$\langle\, c\, \partial c\, \partial^2 c\, \rangle = 1 \tag{2.23}$$

where c is the worldsheet ghost of dimension -1. Since $c\partial c\partial^2 c$ is the vertex operator of $+3$ ghost-number for the Yang-Mills antighost [38], it is natural to use the ansatz that non-vanishing correlation functions in this formalism must also be proportional to the vertex operator for the Yang-Mills antighost. As discussed in the previous subsection, this vertex operator in the pure spinor formalism is

$$V = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta), \tag{2.24}$$

which is the unique state in the BRST cohomology at $+3$ ghost-number. So the zero mode prescription for tree amplitudes in the pure spinor formalism is

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1. \tag{2.25}$$

For later use, it will be convenient to write (2.24) as

$$V = \mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]}\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\delta_1}\theta^{\delta_2}\theta^{\delta_3}\theta^{\delta_4}\theta^{\delta_5}, \tag{2.26}$$

where

$$\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]} \tag{2.27}$$

is a constant Lorentz-invariant tensor and the notation $((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]$ signifies that the tensor is symmetric and γ-matrix traceless (i.e. $\gamma_m^{\alpha_1\alpha_2}\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\ldots\delta_5]} = 0$) in the first three indices, and antisymmetric in the last five indices. This tensor is uniquely defined up to rescaling and can be computed by starting with $\gamma^m_{\alpha_1\delta_1}\gamma^n_{\alpha_2\delta_2}\gamma^p_{\alpha_3\delta_3}(\gamma_{mnp})_{\delta_4\delta_5}$, then symmetrizing in the α indices, antisymmetrizing in the δ indices, and subtracting off the γ-matrix trace in the α indices. Similarly, one can define the tensor

$$(\mathcal{T}^{-1})^{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]} \tag{2.28}$$

by starting with $(\gamma^m)^{\alpha_1\delta_1}(\gamma^n)^{\alpha_2\delta_2}(\gamma^p)^{\alpha_3\delta_3}(\gamma_{mnp})^{\delta_4\delta_5}$ and following the same procedure.

Using the properties of spinors in ten dimensions, it will be possible to prove various identities satisfied by $\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]}$. For example, there are no Lorentz scalars which can be constructed out of four λ's and four θ's, which implies that

$$\delta^{\delta_5}_{((\alpha_1}\mathcal{T}_{\alpha_2\alpha_3\alpha_4))[\delta_1\delta_2\delta_3\delta_4\delta_5]} = 0 \tag{2.29}$$

and that

$$\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]} \neq \mathcal{S}_{((\alpha_1\alpha_2\alpha_3\alpha_4))[\delta_1\delta_2\delta_3\delta_4}\delta^{\alpha_4}_{\delta_5]} \tag{2.30}$$

for any tensor $\mathcal{S}_{((\alpha_1\alpha_2\alpha_3\alpha_4))[\delta_1\delta_2\delta_3\delta_4]}$. Furthermore, there are no Lorentz scalars which can be constructed out of two λ's and six θ's, which implies that

$$\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5}\delta^{\alpha_3}_{\delta_6]} = 0 \tag{2.31}$$

and that

$$\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]} \neq \delta^{\delta_6}_{((\alpha_1}\mathcal{S}_{\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6]} \tag{2.32}$$

for any tensor $\mathcal{S}_{((\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6]}$. Finally, one can check that

$$(\lambda\gamma^q\theta)(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) = 0$$

for any q, which implies that

$$\delta^{\kappa}_{((\alpha_1}\mathcal{T}_{\alpha_2\alpha_3\alpha_4))[\delta_1\delta_2\delta_3\delta_4\delta_5}\gamma^q_{\delta_6]\kappa} = 0. \tag{2.33}$$

Using (2.25), the zero-mode prescription for tree amplitudes is

$$\langle \mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1...\delta_5]}\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\delta_1}...\theta^{\delta_5}\rangle = 1.$$

In other words, suppose that $\mathcal{A} = \langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)\rangle$ is the expression one gets after integrating out the non-zero modes, where $f_{\alpha\beta\gamma}$ is some complicated function of the polarizations and momenta of the external states. Then the scattering amplitude is defined as

$$\mathcal{A} = (\mathcal{T}^{-1})^{((\alpha\beta\gamma))[\delta_1...\delta_5]}\frac{\partial}{\partial\theta^{\delta_1}}...\frac{\partial}{\partial\theta^{\delta_5}}f_{\alpha\beta\gamma}(\theta). \tag{2.34}$$

Using this prescription and the identities of (2.29)-(2.33), it was shown in [14] that on-shell tree amplitudes are gauge-invariant and supersymmetric. And it was shown in [16] and [17] that this tree amplitude prescription agrees with the standard RNS prescription. However, it was unclear how to generalize this prescription to loop amplitudes since it was not derived from functional integration. In the next section, it will be shown how to use picture-changing operators to resolve this problem.

3. Functional Integration and Picture-Changing Operators

As reviewed in section (2.1), the gauge invariance of (2.4) implies that pure spinor ghosts can only appear through the operators λ^α, N_{mn} and J. Correlation functions for the non-zero modes of these operators are easily computed using the OPE's of (2.6). However, after integrating out the non-zero worldsheet modes, one still has to functionally integrate over the worldsheet zero modes. Because λ^α has zero conformal weight and satisfies the pure spinor constraint

$$\lambda\gamma^m\lambda = 0, \tag{3.1}$$

λ^α has 11 independent zero modes on a genus g surface. And because N_{mn} and J have $+1$ conformal weight and are defined from gauge-invariant combinations of w_α, they have $11g$ independent zero modes on a genus g surface. Note that (3.1) implies that $N_{mn} = \frac{1}{2}(w\gamma_{mn}\lambda)$ and $J = w\lambda$ are related by the equation[37]

$$: N^{mn}\lambda^\alpha : \gamma_{m\alpha\beta} - \frac{1}{2} : J\lambda^\alpha : \gamma^n_{\alpha\beta} = 2\gamma^n_{\alpha\beta}\partial\lambda^\alpha \tag{3.2}$$

where the normal-ordered product is defined by $: U^A(z)\lambda^\alpha(z) := \oint dy(y-z)^{-1}U^A(y)\lambda^\alpha(z)$. (The coefficient of the $\partial\lambda^\alpha$ term is determined by computing the double pole of the left-hand side of (3.2) with J.) Just as (3.1) implies that all 16 components of λ^α can be expressed in terms of 11 components, equation (3.2) implies that all 45 components of N^{mn} can be expressed in terms of J and ten components of N^{mn}.

Because of the constraints of (3.1) and (3.2), it is not immediately obvious how to functionally integrate over the pure spinor ghosts. However, as will be shown in the following subsection, there is a natural Lorentz-invariant measure factor for the pure spinor ghosts which can be used to define functional integration.

3.1. Measure factor for pure spinor ghosts

A Lorentz-invariant measure factor for the λ^α zero modes can be obtained by noting that

$$(d^{11}\lambda)^{[\alpha_1\alpha_2\ldots\alpha_{11}]} \equiv d\lambda^{\alpha_1} \wedge d\lambda^{\alpha_2} \wedge \ldots \wedge d\lambda^{\alpha_{11}} \tag{3.3}$$

satisfies the identity

$$\lambda^\beta\gamma^m_{\alpha_1\beta}(d^{11}\lambda)^{[\alpha_1\alpha_2\ldots\alpha_{11}]} = 0 \tag{3.4}$$

because $\lambda\gamma^m d\lambda = 0$. Using the properties of pure spinors, this implies that all $\frac{16!}{5!11!}$ components of $(d^{11}\lambda)^{[\alpha_1...\alpha_{11}]}$ are related to each other by a Lorentz-invariant measure factor $[\mathcal{D}\lambda]$ of +8 ghost number which is defined by

$$(d^{11}\lambda)^{[\alpha_1...\alpha_{11}]} = [\mathcal{D}\lambda] \;\; (\epsilon\mathcal{T})^{[\alpha_1...\alpha_{11}]}_{((\beta_1\beta_2\beta_3))} \lambda^{\beta_1}\lambda^{\beta_2}\lambda^{\beta_3} \tag{3.5}$$

where

$$(\epsilon\mathcal{T})^{[\alpha_1...\alpha_{11}]}_{((\beta_1\beta_2\beta_3))} = \epsilon^{\alpha_1...\alpha_{16}}\mathcal{T}_{((\beta_1\beta_2\beta_3))[\alpha_{12}...\alpha_{16}]}$$

and (3.4) is implied by (3.5) using the identity of (2.33). In other words, for any choice of $[\alpha_1...\alpha_{11}]$, one can define the Lorentz-invariant measure $[\mathcal{D}\lambda]$ by the formula

$$[\mathcal{D}\lambda] = (d^{11}\lambda)^{[\alpha_1...\alpha_{11}]} \; [(\epsilon\mathcal{T})^{[\alpha_1...\alpha_{11}]}_{((\beta_1\beta_2\beta_3))} \lambda^{\beta_1}\lambda^{\beta_2}\lambda^{\beta_3}]^{-1}, \tag{3.6}$$

where there is no sum over $[\alpha_1...\alpha_{11}]$ in (3.6).

One can similarly construct a Lorentz-invariant measure factor for the N^{mn} and J zero modes from

$$(d^{11}N)^{[[m_1n_1][m_2n_2]...[m_{10}n_{10}]]} \equiv dN^{[m_1n_1]} \wedge dN^{[m_2n_2]} \wedge ... \wedge dN^{[m_{10}n_{10}]} \wedge dJ. \tag{3.7}$$

Using the constraint of (3.2) and keeping λ^α fixed while varying N^{mn} and J, one finds that (3.7) satisfies the identity

$$(\lambda\gamma_{m_1})_\alpha (d^{11}N)^{[[m_1n_1][m_2n_2]...[m_{10}n_{10}]]} = 0. \tag{3.8}$$

Using the properties of pure spinors, this implies that all $\frac{45!}{10!35!}$ components of

$$(d^{11}N)^{[[m_1n_1][m_2n_2]...[m_{10}n_{10}]]}$$

are related to each other by a Lorentz-invariant measure factor $[\mathcal{D}N]$ of -8 ghost number which is defined by

$$(d^{11}N)^{[[m_1n_1][m_2n_2]...[m_{10}n_{10}]]} = [\mathcal{D}N] \tag{3.9}$$

$$((\lambda\gamma^{m_1n_1m_2m_3m_4}\lambda)(\lambda\gamma^{m_5n_5n_2m_6m_7}\lambda)(\lambda\gamma^{m_8n_8n_3n_6m_9}\lambda)(\lambda\gamma^{m_{10}n_{10}n_4n_7n_9}\lambda) + \text{permutations})$$

where the permutations are antisymmetric under the exchange of m_j with n_j, and also antisymmetric under the exchange of $[m_jn_j]$ with $[m_kn_k]$. Note that the index structure on the right-hand side of (3.9) has been chosen such the expression is non-vanishing after summing over the permutations.

After using the OPE's of (2.6) to integrate out the non-zero modes of the pure spinor ghosts on a genus g surface, one will obtain an expression

$$\mathcal{A} = \langle f(\lambda, N_1, J_1, N_2, J_2, ..., N_g, J_g) \rangle \tag{3.10}$$

which only depends on the 11 worldsheet zero modes of λ, and on the $11g$ worldsheet zero modes of N and J. Using the Lorentz-invariant measure factors defined in (3.5) and (3.9), the natural definition for functional integration over these zero modes is

$$\mathcal{A} = \int [\mathcal{D}\lambda][\mathcal{D}N_1][\mathcal{D}N_2]...[\mathcal{D}N_g] f(\lambda, N_1, J_1, N_2, J_2, ..., N_g, J_g). \tag{3.11}$$

Note that with this definition, $f(\lambda, N_1, J_1, N_2, J_2, ..., N_g, J_g)$ must carry ghost number $-8+8g$ to give a non-vanishing functional integral, which agrees with the -8 ghost-number anomaly in the OPE of J with T. It will now be shown how the functional integral of (3.11) can be explicitly computed with the help of picture-changing operators.

3.2. Picture-changing operators

As is well-known from the work of Friedan-Martinec-Shenker [6] and Verlinde-Verlinde [2][3], picture-changing operators are necessary in the RNS formalism because of the bosonic (β, γ) ghosts. Since the picture-raising and picture-lowering operators involve the delta functions $\delta(\beta)$ and $\delta(\gamma)$, insertion of these operators in loop amplitudes are needed to absorb the zero modes of the (β, γ) ghosts on a genus g surface.[12] Up to possible surface terms, the amplitudes are independent of the worldsheet positions of these operators since the worldsheet derivatives of the picture-changing operators are BRST-trivial. The surface terms come from pulling the BRST operator through the b ghosts to give total derivatives in the worldsheet moduli. If the correlation function diverges near the boundary of moduli space, these surface terms can give finite contributions which need to be treated carefully.

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[12] In the RNS formalism, it is convenient to bosonize the (β, γ) ghosts as $\beta = \partial\xi e^{-\phi}$ and $\gamma = \eta e^{\phi}$ since the spacetime supersymmetry generator involves a spin field constructed for the chiral boson ϕ. The delta functions $\delta(\beta)$ and $\delta(\gamma)$ can then be expressed in terms of ϕ as $\delta(\beta) = e^{\phi}$ and $\delta(\gamma) = e^{-\phi}$. However, in the pure spinor formalism, there is no advantage to performing such a bosonization since all operators can be expressed directly in terms of λ^α, N^{mn} and J. Since functional integration over the ϕ chiral boson can give rise to unphysical poles in the correlation functions, the fact that all operators in the pure spinor formalism can be expressed in terms of $(\lambda^\alpha, N^{mn}, J)$ implies that there are no unphysical poles in pure spinor correlation functions.

As will now be shown, functional integration over the bosonic ghosts in the pure spinor formalism also requires picture-changing operators with similar properties to those of the RNS formalism. However, since the correlation functions in this formalism do not diverge near the boundary of moduli space, there are no subtleties due to surface terms.

To absorb the zero modes of λ^α, N_{mn} and J, picture-changing operators in the pure spinor formalism will involve the delta-functions $\delta(C_\alpha\lambda^\alpha)$, $\delta(B_{mn}N^{mn})$ and $\delta(J)$ where C_α and B_{mn} are constant spinors and antisymmetric tensors. Although these constant spinors and tensors are needed for the construction of picture-changing operators, it will be shown that scattering amplitudes are independent of the choice of C_α and B_{mn}, so Lorentz invariance is preserved. As will be discussed later, this Lorentz invariance can be made manifest by integrating over all choices of C_α and B_{mn}. Note that the use of constant spinors and tensors in picture-changing operators is unrelated to the pure spinor constraint, and is necessary whenever the bosonic ghosts are not Lorentz scalars.

As in the RNS formalism, the picture-changing operators will be BRST-invariant with the property that their worldsheet derivative is BRST-trivial. A "picture-lowering" operator Y_C with these properties is

$$Y_C = C_\alpha\theta^\alpha\delta(C_\beta\lambda^\beta) \tag{3.12}$$

where C_α is any constant spinor. Note that $QY_C = (C_\alpha\lambda^\alpha)\delta(C_\beta\lambda^\beta) = 0$ and

$$\partial Y_C = (C\partial\theta)\delta(C\lambda) + (C\theta)(C\partial\lambda)\partial\delta(C\lambda) = Q[(C\partial\theta)(C\theta)\partial\delta(C\lambda)] \tag{3.13}$$

where $\partial\delta(x) \equiv \frac{\partial}{\partial x}\delta(x)$ is defined using the usual rules for derivatives of delta functions, e.g. $x\partial\delta(x) = -\delta(x)$.[13]

Although Y_C is not spacetime-supersymmetric, its supersymmetry variation is BRST-trivial since

$$q_\alpha Y_C = C_\alpha\delta(C\lambda) = -C_\alpha(C\lambda)\partial\delta(C\lambda) = Q[-C_\alpha(C\theta)\partial\delta(C\lambda)]. \tag{3.14}$$

Similarly, Y_C is not Lorentz invariant, but its Lorentz variation is BRST-trivial since

$$M^{mn}Y_C = \frac{1}{2}(C\gamma^{mn}\theta)\delta(C\lambda) + \frac{1}{2}(C\theta)(C\gamma^{mn}\lambda)\partial\delta(C\lambda) = Q[\frac{1}{2}(C\gamma^{mn}\theta)(C\theta)\partial\delta(C\lambda)]. \tag{3.15}$$

[13] Throughout this paper, the symbol ∂ will denote the worldsheet derivative $\frac{\partial}{\partial z}$ except when ∂ acts on a delta function. When acting on a delta function, $\partial\delta(x)$ will denote $\frac{\partial}{\partial x}\delta(x)$.

So different choices of C_α only change Y_C by a BRST-trivial quantity, and any on-shell amplitude computations involving insertions of Y_C will be Lorentz invariant and spacetime supersymmetric up to possible surface terms. The fact that Lorentz invariance is preserved only up to surface terms is unrelated to the pure spinor constraint, and is caused by the bosonic ghosts not being Lorentz scalars.

One can also construct BRST-invariant operators involving $\delta(B^{mn}N_{mn})$ and $\delta(J)$ with the property that their worldsheet derivative is BRST-trivial. These "picture-raising" operators will be called Z_B and Z_J and are defined by

$$Z_B = \frac{1}{2}B_{mn}(\lambda\gamma^{mn}d)\delta(B^{pq}N_{pq}), \quad Z_J = (\lambda^\alpha d_\alpha)\delta(J), \tag{3.16}$$

where B_{mn} is a constant antisymmetric tensor. To eliminate the need for normal-ordering in Z_B, it will be convenient to choose B_{mn} such that it satisfies

$$B_{mn}B_{pq}(\gamma^{mn}\gamma^p)_\alpha{}^\beta = 0. \tag{3.17}$$

(To give a concrete example, B_{mn} satisfies (3.17) if its only non-zero components are in the directions $B_{13} = iB_{23} = -B_{24} = iB_{14}$.) With this choice, $(\lambda\gamma^{mn}d)B_{mn}$ has no pole with $B^{pq}N_{pq}$, and therefore $(\lambda\gamma^{mn}d)B_{mn}$ has no pole with $\delta(B^{pq}N_{pq})$.

Since $\lambda^\alpha d_\alpha$ has a pole with $\delta(J)$, it naively appears that Z_J needs to be regularized. However, $\lambda^\alpha d_\alpha$ is the BRST current which has no poles anywhere else on the surface. Since Z_J will only be needed on surfaces of non-zero genus, and since any function with a single pole on such surfaces must be a constant function, $\lambda^\alpha d_\alpha$ has no pole with $\delta(J)$ and therefore Z_J does not need to be regularized.

Z_B and Z_J satisfy the properties of picture-changing operators since

$$QZ_B = -\frac{1}{4}B_{mn}B_{pq}(\lambda\gamma^{mn}d)(\lambda\gamma^{pq}d)\partial\delta(BN) - \frac{1}{2}B_{mn}\Pi_p(\lambda\gamma^{mn}\gamma^p\lambda)\delta(BN) = 0, \tag{3.18}$$

$$QZ_J = (\lambda_\alpha d^\alpha)(\lambda_\beta d^\beta)\partial\delta(J) - \Pi_m(\lambda\gamma^m\lambda)\delta(J) = 0,$$

$$\partial Z_B = \frac{1}{2}B_{mn}\partial(\lambda\gamma^{mn}d)\delta(BN) + \frac{1}{2}B_{mn}(\lambda\gamma^{mn}d)B_{pq}\partial N^{pq}\partial\delta(BN) = Q[B_{pq}\partial N^{pq}\delta(BN)],$$

$$\partial Z_J = \partial(\lambda d)\delta(J) + (\lambda d)\partial J\partial\delta(J) = Q[-\partial J\delta(J)].$$

Furthermore, Z_B and Z_J are manifestly spacetime supersymmetric and the Lorentz transformation of Z_B is BRST-trivial since

$$M^{mn}Z_B = \eta^{p[m}\delta_r^{n]}[B_{pq}(\lambda\gamma^{qr}d)\delta(BN) + B_{st}(\lambda\gamma^{st}d)B_{pq}N^{qr}\partial\delta(BN)] \tag{3.19}$$

$$= Q[2\eta^{p[m}\delta_r^{n]}B_{pq}N^{qr}\delta(BN)].$$

So different choices of B_{mn} only change Z_B by a BRST-trivial quantity.

With these picture-changing operators, the pure spinor measure factors of subsection (3.1) can be used to compute arbitrary loop amplitudes with functional integration methods. But before discussing loop amplitudes, it will be useful to show how these functional integration methods reproduce the tree amplitude prescription of subsection (2.3).

3.3. Functional integration computation of tree amplitude

For tree amplitudes, λ^α has eleven zero modes so one needs to insert eleven picture-lowering operators $Y_{C_1}(y_1)...Y_{C_{11}}(y_{11})$ into the correlation function where the choices of C_I and y_I for $I = 1$ to 11 are arbitrary. It will now be shown that functional integration with these insertions reproduces the correct tree amplitude prescription. Using the notation of subsection (2.3), the N-point open string tree amplitude is computed by the correlation function

$$\mathcal{A} = \langle\ V_1(z_1)\ V_2(z_2)\ V_3(z_3)\ \int dz_4 U_4(z_4)...\int dz_N U_N(z_N)\ Y_{C_1}(y_1)...Y_{C_{11}}(y_{11})\rangle \qquad (3.20)$$

where V and U are the unintegrated and integrated vertex operators and $Y_{C_I}(y_I) = C_{I\alpha}\theta^\alpha(y_I)\delta(C_I\lambda(y_I))$.

To compare with the prescription of (2.20), it is convenient to fix (z_1, z_2, z_3) at finite points on the worldsheet and to insert all eleven picture-lowering operators at $y_I = \infty$. With this choice, there are no contributions from the OPE's of the picture-lowering operators with the N vertex operators. Also, there are no singular OPE's between the picture-lowering operators since $\delta(C_1\lambda)$ has a pole with $\delta(C_2\lambda)$ only when $C_{1\alpha}$ is proportional to $C_{2\alpha}$, which implies that $(C_1\theta)$ has a zero with $(C_2\theta)$. After integrating over the non-zero modes of the worldsheet fields, one is left with the expression

$$\mathcal{A} = \langle\ \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)(C_1\theta)...(C_{11}\theta)\delta(C_1\lambda)...\delta(C_{11}\lambda)\ \rangle \qquad (3.21)$$

where $f_{\alpha\beta\gamma}(\theta)$ is the same function as in the computation of (2.34). To integrate over the θ^α and λ^α zero modes, use the standard $\int d^{16}\theta$ measure factor and the pure spinor measure factor of (3.5) to obtain

$$\mathcal{A} = \int d^{16}\theta \int [\mathcal{D}\lambda]\ \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)(C_1\theta)...(C_{11}\theta)\delta(C_1\lambda)...\delta(C_{11}\lambda) \qquad (3.22)$$

$$= \int d^{16}\theta (\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]} \int d\lambda^{\rho_1}...d\lambda^{\rho_{11}} f_{\alpha\beta\gamma}(\theta)(C_1\theta)...(C_{11}\theta)\delta(C_1\lambda)...\delta(C_{11}\lambda).$$

In general,

$$\int d\lambda^{\rho_1}...d\lambda^{\rho_{11}} \delta(C_1\lambda)...\delta(C_{11}\lambda) \tag{3.23}$$

is a complicated function of C_I because of the Jacobian coming from expressing λ^ρ in terms of $(C_I\lambda)$. However, since the Lorentz variation of Y_C is BRST-trivial, the amplitude is independent (up to possible surface terms) of the choice of C_I. This implies that if one integrates (3.22) over all possible choices for C_I with a measure factor $[\mathcal{D}C]$ satisfying $\int [\mathcal{D}C] = 1$, the amplitude is unchanged. Note that (3.22) is manifestly invariant under rescalings of C_I, so C_I can be interpreted as a projective coordinate.

So one can express the amplitude in Lorentz-covariant form as

$$\mathcal{A} = (\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]} \int d^{16}\theta \ \theta^{\kappa_1}...\theta^{\kappa_{11}} f_{\alpha\beta\gamma}(\theta) \tag{3.24}$$

$$\int [\mathcal{D}C] \int d\lambda^{\rho_1}...d\lambda^{\rho_{11}} C_{1\kappa_1}...C_{11\kappa_{11}} \delta(C_1\lambda)...\delta(C_{11}\lambda).$$

By Lorentz invariance,

$$\int [\mathcal{D}C] \int d\lambda^{\rho_1}...d\lambda^{\rho_{11}} C_{1\kappa_1}...C_{11\kappa_{11}} \delta(C_1\lambda)...\delta(C_{11}\lambda) = c\delta^{[\rho_1}_{\kappa_1}...\delta^{\rho_{11}]}_{\kappa_{11}} \tag{3.25}$$

where c is a normalization factor which is determined from

$$\int [\mathcal{D}C] \int (C_{1\rho_1} d\lambda^{\rho_1})...(C_{11\rho_{11}} d\lambda^{\rho_{11}}) \delta(C_1\lambda)...\delta(C_{11}\lambda) = 1. \tag{3.26}$$

So

$$\mathcal{A} = c(\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\kappa_1...\kappa_{11}]} \int d^{16}\theta \ \theta^{\kappa_1}...\theta^{\kappa_{11}} f_{\alpha\beta\gamma}(\theta), \tag{3.27}$$

which agrees with the tree amplitude prescription of (2.34) up to a constant normalization factor.

Note that the above computation can be easily generalized to correlation functions where the picture-lowering operators are not at $y_I = \infty$. In this case, one can get factors such as $\partial\delta(C\lambda)$ from OPE's between the picture-lowering operators and the vertex operators. However, since the amplitude is guaranteed to be independent of C_I, one can use a similar argument to trivially perform the functional integration over the pure spinor ghosts.

Using Lorentz invariance and symmetry properties, the previous prescription for integrating over λ^α zero modes can be generalized to a prescription for evaluating $\langle f(\lambda, C_I)\rangle$ where

$$f(\lambda, C_I) = h(\lambda, C_I) \prod_{I=1}^{11} \partial^{K_I} \delta(C_I \lambda) \tag{3.28}$$

and $h(\lambda, C_I)$ is a polynomial depending on λ^α and $C_{I\alpha}$ as

$$h(\lambda, C_I) = (\lambda)^{3+\sum_{I=1}^{11} K_I} \prod_{I=1}^{11} (C_I)^{K_I+1}.$$

The manifestly Lorentz-covariant prescription is

$$\langle f(\lambda, C_I)\rangle = \int [\mathcal{D}C] \int [\mathcal{D}\lambda]\ f(\lambda, C_I) \tag{3.29}$$

$$= c'(\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \ldots \rho_{11}]} \frac{\partial}{\partial\lambda^\alpha} \frac{\partial}{\partial\lambda^\beta} \frac{\partial}{\partial\lambda^\gamma} \frac{\partial}{\partial C_{1\rho_1}} \cdots \frac{\partial}{\partial C_{11\rho_{11}}} \prod_{I=1}^{11} \left(\frac{\partial}{\partial\lambda^\delta} \frac{\partial}{\partial C_{I\delta}}\right)^{K_I} h(\lambda, C_I),$$

where c' is a proportionality constant which can be computed as in (3.26).

As will be shown in section 5, similar methods can be used to perform functional integration over the N^{mn} and J zero modes in loop amplitudes. However, before discussing these loop amplitudes, it will be necessary to first construct an appropriate b ghost.

4. Construction of b Ghost

To compute g-loop amplitudes, the usual string theory prescription requires the insertion of $(3g-3)$ b ghosts of -1 ghost-number which satisfy

$$\{Q, b(u)\} = T(u) \tag{4.1}$$

where T is the stress tensor of (2.7). After integrating $b(u)$ with a Beltrami differential $\mu_P(u)$ for $P = 1$ to $3g-3$, the BRST variation of $b(u)$ generates a total derivative with respect to the Teichmuller parameter τ_P associated to the Beltrami differential μ_P. But since w_α can only appear in gauge-invariant combinations of zero ghost number, there are no operators of negative ghost number in the pure spinor formalism, so one cannot construct such a b ghost. Nevertheless, as will now be shown, the picture-raising operator

$$Z_B = \frac{1}{2} B_{mn} (\lambda\gamma^{mn} d) \delta(BN)$$

can be used to construct a suitable substitute for the b ghost in non-zero picture.

Since genus g amplitudes also require $10g$ insertions of $Z_B(z)$, one can combine $(3g-3)$ insertions of $Z_B(z)$ with the desired insertions of the $b(u)$ ghost and look for a non-local operator $\widetilde{b}_B(u,z)$ which satisfies

$$\{Q, \widetilde{b}_B(u,z)\} = T(u) Z_B(z). \tag{4.2}$$

Note that Z_B carries $+1$ ghost-number, so $\widetilde{b}_B$ carries zero ghost number. And (4.2) implies that integrating $\widetilde{b}(u,z)$ with the Beltrami differential $\mu_P(u)$ has the same properties as integrating $b(u)$ with $\mu_P(u)$ in the presence of a picture-raising operator $Z_B(z)$.

Using

$$Z_B(z) = Z_B(u) + \int_u^z dv \partial Z_B(v) = Z_B(u) + \int_u^z dv \{Q, B_{pq} \partial N^{pq}(v) \delta(BN(v))\},$$

one can define

$$\widetilde{b}_B(u,z) = b_B(u) + T(u) \int_u^z dv B_{pq} \partial N^{pq}(v) \delta(BN(v)) \tag{4.3}$$

where $b_B(u)$ is a local operator satisfying[14]

$$\{Q, b_B(u)\} = T(u) Z_B(u). \tag{4.4}$$

4.1. Chain of operators

To construct b_B satisfying (4.4), it is useful to first construct a chain of operators which are related to the stress tensor T through BRST transformations. Although there is no b operator of -1 ghost-number satisfying $\{Q, b\} = T$, there is an operator G^α of zero ghost-number satisfying

$$\{Q, G^\alpha\} = \lambda^\alpha T. \tag{4.5}$$

The existence of G^α is guaranteed since $\lambda^\alpha T$ is a BRST-invariant operator of $+1$ ghost number and $+2$ conformal weight, and the BRST cohomology at $+1$ ghost number is non-trivial only at zero conformal weight. One finds that[17]

$$G^\alpha = \frac{1}{2}\Pi^m (\gamma_m\, d)^\alpha - \frac{1}{4} N_{mn} (\gamma^{mn} \partial\theta)^\alpha - \frac{1}{4} J \partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha \tag{4.6}$$

---

[14] A similar picture-raised version of the b ghost appears in the N=4 topological description of the superstring [40] as the $\widetilde{G}^-$ generator. Since the pure spinor formalism can be related to the N=4 topological description through the twistor approach of [41] [42], it would be interesting to try to relate b_B with $\widetilde{G}^-$ using the approach of [41].

where the $\partial^2\theta^\alpha$ term comes from normal-ordering. Note that if one ignores this normal-ordering contribution, all terms in G^α carry "engineering dimension" $\frac{5}{2}$ where $[\lambda^\alpha, \theta^\alpha, x^m, p_\alpha, w_\alpha]$ are defined to carry engineering dimension $[0, \frac{1}{2}, 1, \frac{3}{2}, 2]$. Furthermore, one can verify that G^α is a primary field of $+2$ conformal weight.

Since

$$Q(\lambda^\alpha G^\beta) = \lambda^\alpha \lambda^\beta T \tag{4.7}$$

is symmetric and γ-matrix traceless (i.e. $Q(\lambda^{[\alpha}G^{\beta]}) = Q(\lambda\gamma^m G) = 0$), cohomology arguments imply there exists an operator $H^{\alpha\beta}$ which satisfies

$$Q(H^{\alpha\beta}) = \lambda^\alpha G^\beta + g^{((\alpha\beta))} \tag{4.8}$$

where $g^{((\alpha\beta))}$ is some symmetric γ-matrix traceless operator.[15] Note that (4.8) only determines $H^{\alpha\beta}$ up to the gauge transformation

$$\delta H^{\alpha\beta} = \Omega^{((\alpha\beta))} \tag{4.9}$$

where $\Omega^{((\alpha\beta))}$ is any symmetric γ-matrix traceless operator. For example, one can choose $\Omega^{((\alpha\beta))}$ such that $H^{((\alpha\beta))} = 0$, in which case (4.8) is solved by

$$H^{\alpha\beta} = \frac{1}{16}\gamma_m^{\alpha\beta}(N^{mn}\Pi_n - \frac{1}{2}J\Pi^m + 2\partial\Pi^m) + \frac{1}{384}\gamma_{mnp}^{\alpha\beta}(d\gamma^{mnp}d + 24N^{mn}\Pi^p) \tag{4.10}$$

where $g^{((\alpha\beta))} = -\frac{1}{3840}\gamma_{mnpqr}^{\alpha\beta}(\lambda\gamma^{mnpqr}G)$. One can check that $H^{\alpha\beta}$ is a primary field of conformal weight $+2$ and that, if one ignores the normal-ordering term proportional to $\partial\Pi^m$, all terms in $H^{\alpha\beta}$ carry $+3$ engineering dimension.[16]

The next link in the chain of operators is constructed by noting that

$$Q(\lambda^\alpha H^{\beta\gamma}) = \lambda^\alpha\lambda^\beta G^\gamma + \lambda^\alpha g^{((\beta\gamma))}, \tag{4.11}$$

which implies using similar cohomology arguments as before that there exists an operator $K^{\alpha\beta\gamma}$ which satisfies

$$Q(K^{\alpha\beta\gamma}) = \lambda^\alpha H^{\beta\gamma} + h_1^{((\alpha\beta))\gamma} + h_2^{\alpha((\beta\gamma))} \tag{4.12}$$

[15] Since $(\lambda^\alpha G^\beta - \lambda^{((\alpha}G^{\beta))})$ is a BRST-invariant operator of $+1$ ghost-number and $+2$ conformal weight, it is guaranteed that $(\lambda^\alpha G^\beta - \lambda^{((\alpha}G^{\beta))}) = Q(H^{\alpha\beta} - H^{((\alpha\beta))})$ for some $H^{\alpha\beta}$. Defining $g^{((\alpha\beta))} = -\lambda^{((\alpha}G^{\beta))} + Q(H^{((\alpha\beta))})$, one recovers (4.8).

[16] It is interesting to note that the $[T, G^\alpha, H^{\alpha\beta}]$ operators closely resemble the $[A, B^\alpha, C^{mnp}]$ constraints of Siegel [35] for quantization of the superparticle and superstring.

where $h_1^{((\alpha\beta))\gamma}$ is some operator which is symmetric γ-matrix traceless in its first two indices, and $h_2^{\alpha((\beta\gamma))}$ is some operator which is symmetric γ-matrix traceless in its last two indices. Note that (4.12) only determines $K^{\alpha\beta\gamma}$ up to the gauge transformation

$$\delta K^{\alpha\beta\gamma} = \Omega_1^{((\alpha\beta))\gamma} + \Omega_2^{\alpha((\beta\gamma))}. \tag{4.13}$$

From its $\frac{7}{2}$ engineering dimension (ignoring normal-ordering terms) and its +2 conformal weight, one can deduce that

$$K^{\alpha\beta\gamma} = c_{1mn}^{\alpha\beta\gamma\rho} N^{mn} d_\rho + c_2^{\alpha\beta\gamma\rho} J d_\rho + c_3^{\alpha\beta\gamma\rho} \partial d_\rho, \tag{4.14}$$

however, the coefficients $[c_{1mn}^{\alpha\beta\gamma\rho}, c_2^{\alpha\beta\gamma\rho}, c_3^{\alpha\beta\gamma\rho}]$ have not yet been computed.[17]

Finally, the last link in the chain of operators is constructed by noting that

$$Q(\lambda^\alpha K^{\beta\gamma\delta}) = \lambda^\alpha \lambda^\beta H^{\gamma\delta} + \lambda^\alpha h_1^{((\beta\gamma))\delta} + \lambda^\alpha h_2^{\beta((\gamma\delta))}, \tag{4.15}$$

which implies that there exists an operator $L^{\alpha\beta\gamma\delta}$ which satisfies

$$Q(L^{\alpha\beta\gamma\delta}) = \lambda^\alpha K^{\beta\gamma\delta} + k_1^{((\alpha\beta))\gamma\delta} + k_2^{\alpha((\beta\gamma))\delta} + k_3^{\alpha\beta((\gamma\delta))}, \tag{4.16}$$

where $[k_1^{((\alpha\beta))\gamma\delta}, k_2^{\alpha((\beta\gamma))\delta}, k_3^{\alpha\beta((\gamma\delta))}]$ are operators which are symmetric γ-matrix traceless in their first two, middle two, or last two indices. As before, (4.16) only determines $L^{\alpha\beta\gamma\delta}$ up to the gauge transformation

$$\delta L^{\alpha\beta\gamma\delta} = \Omega_1^{((\alpha\beta))\gamma\delta} + \Omega_2^{\alpha((\beta\gamma))\delta} + \Omega_3^{\alpha\beta((\gamma\delta))}. \tag{4.17}$$

Since $L^{\alpha\beta\gamma\delta}$ carries +4 engineering dimension (ignoring normal-ordering terms) and +2 conformal weight, it has the form

$$L^{\alpha\beta\gamma\delta} = c_{4mnpq}^{\alpha\beta\gamma\delta} N^{mn} N^{pq} + c_{5mn}^{\alpha\beta\gamma\delta} J N^{mn} + c_6^{\alpha\beta\gamma\delta} JJ + c_{7mn}^{\alpha\beta\gamma\delta} \partial N^{mn} + c_8^{\alpha\beta\gamma\delta} \partial J, \tag{4.18}$$

where the coefficients in (4.18) have not yet been computed.

To show that $L^{\alpha\beta\gamma\delta}$ is the last link in the chain of operators, note that there are no supersymmetric primary fields of +2 conformal weight which carry engineering dimension greater than four. So if one tries to define an operator $M^{\alpha\beta\gamma\delta}$ satisfying

$$Q(M^{\alpha\beta\gamma\delta\rho}) = \lambda^\alpha L^{\beta\gamma\delta\rho} + l_1^{((\alpha\beta))\gamma\delta\rho} + l_2^{\alpha((\beta\gamma))\delta\rho} + l_3^{\alpha\beta((\gamma\delta))\rho} + l_4^{\alpha\beta\gamma((\delta\rho))} \tag{4.19}$$

[17] For this type of computation, it would be very helpful to have a computer code designed to handle manipulations of pure spinors.

for some $[l_1^{((\alpha\beta))\gamma\delta\rho}, l_2^{\alpha((\beta\gamma))\delta\rho}, l_3^{\alpha\beta((\gamma\delta))\rho}, l_4^{\alpha\beta\gamma((\delta\rho))}]$, one finds that $M^{\alpha\beta\gamma\delta\rho}$ must vanish. This implies that

$$\lambda^\alpha L^{\beta\gamma\delta\rho} = -l_1^{((\alpha\beta))\gamma\delta\rho} - l_2^{\alpha((\beta\gamma))\delta\rho} - l_3^{\alpha\beta((\gamma\delta))\rho} - l_4^{\alpha\beta\gamma((\delta\rho))}, \quad (4.20)$$

which implies that

$$L^{\alpha\beta\gamma\delta} = \lambda^\alpha S^{\beta\gamma\delta} + s_1^{((\alpha\beta))\gamma\delta} + s_2^{\alpha((\beta\gamma))\delta} + s_3^{\alpha\beta((\gamma\delta))} \quad (4.21)$$

for some $S^{\beta\gamma\delta}$ and $[s_1^{((\alpha\beta))\gamma\delta}, s_2^{\alpha((\beta\gamma))\delta}, s_3^{\alpha\beta((\gamma\delta))}]$. For the following subsection, it will be useful to note that (4.21) and (4.16) imply that

$$Q(S^{\beta\gamma\delta}) = K^{\beta\gamma\delta} + \lambda^\beta T^{\gamma\delta} + t_1^{((\beta\gamma))\delta} + t_2^{\beta((\gamma\delta))} \quad (4.22)$$

for some $[T^{\gamma\delta}, t_1^{((\beta\gamma))\delta}, t_2^{\beta((\gamma\delta))}]$.

Note that $S^{\beta\gamma\delta}$ has ghost-number -1, so it will depend on w_α in combinations which are not invariant under the gauge transformation of (2.4). However, since $L^{\alpha\beta\gamma\delta}$ only involves gauge-invariant combinations of w_α, the change in $S^{\beta\gamma\delta}$ under (2.4) must be of the form

$$\delta S^{\beta\gamma\delta} = \lambda^\beta \Sigma^{\gamma\delta} + \rho_1^{((\beta\gamma))\delta} + \rho_2^{\beta((\gamma\delta))} \quad (4.23)$$

for some $[\Sigma^{\gamma\delta}, \rho_1^{((\beta\gamma))\delta}, \rho_2^{\beta((\gamma\delta))}]$ in order that the change in $S^{\beta\gamma\delta}$ can be cancelled in (4.21) by shifting

$$\delta s_1^{((\alpha\beta))\gamma\delta} = -\lambda^\alpha\lambda^\beta\Sigma^{\gamma\delta}, \quad \delta s_2^{\alpha((\beta\gamma))\delta} = -\lambda^\alpha \rho_1^{((\beta\gamma))\delta}, \quad \delta s_3^{\alpha\beta((\gamma\delta))} = -\lambda^\alpha \rho_2^{\beta((\gamma\delta))}. \quad (4.24)$$

4.2. Construction of b_B

Since $[Q, TZ_B] = 0$ and TZ_B has $+1$ ghost-number and $+2$ conformal weight, cohomology arguments[18] suggest one can find an operator b_B satisfying $\{Q, b_B\} = TZ_B$. Although the structure of b_B will be complicated, one can construct b_B iteratively using the operators $[T, G^\alpha, H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\delta}]$ of the previous subsection. To construct b_B, first note that

$$TZ_B = \frac{1}{2}T(\lambda B d)\delta(BN) = \frac{1}{2}\{Q, G^\alpha\}(Bd)_\alpha \delta(BN) \quad (4.25)$$

[18] At zero picture and $+1$ ghost-number, the BRST cohomology is trivial for states of nonzero conformal weight. It is expected that this is also true at nonzero picture, however, this has not yet been verified.

$$= \frac{1}{2}\{Q, (G\gamma^{mn}d)B_{mn}\delta(BN)\} + \frac{1}{2}G^\alpha[Q, (Bd)_\alpha\delta(BN)]$$

$$= \{Q, b_B^{(1)}\} + \frac{1}{2}G^\alpha[-(\gamma^{mn}\gamma^p\lambda)_\alpha B_{mn}\Pi_p\delta(BN) - \frac{1}{2}(Bd)_\alpha(\lambda Bd)\partial\delta(BN)]$$

where

$$b_B^{(1)} = \frac{1}{2}(G\gamma^{mn}d)B_{mn}\delta(BN), \tag{4.26}$$

$(Bd)_\alpha \equiv (\gamma^{mn}d)_\alpha B_{mn}$, $(\lambda Bd) \equiv (\lambda\gamma^{mn}d)B_{mn}$, and normal-ordering contributions are being ignored.

So one now needs to find an operator $b_B - b_B^{(1)}$ which satisfies

$$\{Q, b_B - b_B^{(1)}\} = -G^\alpha[\frac{1}{2}(\gamma^{mn}\gamma^p\lambda)_\alpha B_{mn}\Pi_p\delta(BN) + \frac{1}{4}(Bd)_\alpha(\lambda Bd)\partial\delta(BN)] \tag{4.27}$$

$$= -\{Q, H^{\beta\alpha}[\frac{1}{2}(\gamma^p\gamma^{nm})_{\beta\alpha}\Pi_p B_{mn}\delta(BN) + \frac{1}{4}(Bd)_\alpha(Bd)_\beta\partial\delta(BN)]\}$$

$$+H^{\beta\alpha}\{Q, \frac{1}{2}(\gamma^p\gamma^{nm})_{\beta\alpha}\Pi_p B_{mn}\delta(BN) + \frac{1}{4}(Bd)_\alpha(Bd)_\beta\partial\delta(BN)\}$$

$$= \{Q, b_B^{(2)}\} + H^{\beta\alpha}[\frac{1}{2}(\gamma^p\gamma^{nm})_{\beta\alpha}(\lambda\gamma_p\partial\theta)B_{mn}\delta(BN) + \frac{1}{4}(\gamma^p\gamma^{nm})_{\beta\alpha}\Pi_p B_{mn}(\lambda Bd)\partial\delta(BN)$$

$$+\frac{1}{4}(Bd)_{[\alpha}(\gamma_{mn}\gamma_p)_{\beta]}B^{mn}\Pi^p\partial\delta(BN) + \frac{1}{8}(Bd)_\alpha(Bd)_\beta(\lambda Bd)\partial^2\delta(BN)]$$

where

$$b_B^{(2)} = -H^{\beta\alpha}[\frac{1}{2}(\gamma^p\gamma^{nm})_{\beta\alpha}\Pi_p B_{mn}\delta(BN) + \frac{1}{4}(Bd)_\alpha(Bd)_\beta\partial\delta(BN)]. \tag{4.28}$$

One now continues this procedure two more stages to construct $b_B^{(3)}$ using $K^{\alpha\beta\gamma}$, and to construct $b_B^{(4)}$ using $L^{\alpha\beta\gamma\delta}$ and $S^{\beta\gamma\delta}$. Using the properties of (4.21) and (4.22), one can verify that the procedure stops here and, ignoring normal-ordering contributions,

$$b_B = b_B^{(1)} + b_B^{(2)} + b_B^{(3)} + b_B^{(4)} \tag{4.29}$$

where $b_B^{(1)}$ and $b_B^{(2)}$ are given in (4.26) and (4.28), and

$$b_B^{(3)} = \frac{1}{2}K^{\gamma\beta\alpha}[(\gamma^p\gamma^{mn})_{\beta\alpha}(\gamma_p\partial\theta)_\gamma B_{mn}\delta(BN)+$$

$$+\frac{1}{2}(\gamma^p\gamma^{nm})_{\beta\alpha}(Bd)_\gamma\Pi_p B_{mn}\partial\delta(BN) + \frac{1}{2}(\gamma^p\gamma^{nm})_{\gamma[\beta}(Bd)_{\alpha]}\Pi_p B_{mn}\partial\delta(BN)$$

$$+\frac{1}{4}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma\partial^2\delta(BN)],$$

$$b_B^{(4)} = \frac{1}{2} S^{\gamma\beta\alpha} (\gamma^p \gamma^{nm})_{\beta\alpha} (\gamma_p \partial\lambda)_\gamma B_{mn} \delta(BN)$$

$$+\frac{1}{4} L^{\delta\gamma\beta\alpha} [\ (\ (\gamma^p\gamma^{nm})_{\beta\alpha} (Bd)_{[\delta} (\gamma_p \partial\theta)_{\gamma]} - (\gamma^p\gamma^{nm})_{\gamma[\beta} (Bd)_{\alpha]} (\gamma_p\partial\theta)_\delta\ )\ B_{mn} \partial\delta(BN)$$

$$-(\ (\gamma^s\gamma^{rq})_{\delta[\gamma} (\gamma^p\gamma^{nm})_{\beta]\alpha} + (\gamma^s\gamma^{rq})_{\delta\alpha} (\gamma^p\gamma^{nm})_{\gamma\beta}\ )\ \Pi_p B_{mn} \Pi_s B_{qr} \partial\delta(BN)$$

$$-\frac{1}{2}(\ (\gamma^p\gamma^{nm})_{\beta\alpha} (Bd)_\gamma (Bd)_\delta + (\gamma^p\gamma^{nm})_{\gamma[\beta} (Bd)_{\alpha]} (Bd)_\delta$$

$$+\frac{1}{2} (\gamma^p\gamma^{nm})_{\delta[\alpha} (Bd)_\beta (Bd)_{\gamma]}\ )\ \Pi_p B_{mn} \partial^2 \delta(BN)$$

$$-\frac{1}{4} (Bd)_\alpha (Bd)_\beta (Bd)_\gamma (Bd)_\delta \partial^3 \delta(BN)\ ].$$

Although b_B of (4.29) is a complicated operator, it has certain simple properties which will be useful to point out. Firstly, b_B is invariant under the gauge transformations of (4.9), (4.13), (4.17) and (4.23) for $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$, $L^{\alpha\beta\gamma\delta}$ and $S^{\beta\gamma\delta}$. Secondly, all terms in b_B have +2 conformal weight (where $\delta(BN)$ has -1 conformal weight). Thirdly, if one ignores normal-ordering contributions[19], all terms in b_B have +4 engineering dimension where $[\lambda^\alpha, \theta^\alpha, x^m, d_\alpha, w_\alpha]$ carry engineering dimension $[0, \frac{1}{2}, 1, \frac{3}{2}, 2]$ and $\delta(BN)$ is defined to carry zero engineering dimension[20]. Fourthly, all terms in b_B are manifestly spacetime supersymmetric. And finally, although b_B is not Lorentz-invariant, its Lorentz transformation only affects the scattering amplitude by a surface term.

To verify this last statement, note that under Lorentz transformations generated by M^{mn}, (3.19) implies that $M^{mn} Z_B = Q\Lambda_B^{mn}$ where

$$\Lambda_B^{mn} = 2\eta^{p[m} \eta_r^{n]} B_{pq} N^{qr} \delta(BN). \tag{4.30}$$

Since $\{Q, b_B\} = TZ_B$, this implies that

$$M^{mn} b_B = T\Lambda_B^{mn} + Q\Omega_B^{mn}$$

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[19] Since terms coming from normal-ordering carry engineering dimension less than +4, they will not contribute to the scattering amplitudes computed in section 6. However, for more general amplitude computations, one will need to include contributions from the normal-ordering terms in b_B.

[20] Although it might seem more natural to define $\delta(BN)$ to carry -2 engineering dimension, it will be more convenient for our purposes to define $\delta(BN)$ to be dimensionless.

for some Ω_B^{mn}. So using (4.3),

$$M^{mn}\widetilde{b}_B(u,z) = M^{mn}b_B(u) + T(u)\int_u^z dv M^{mn}(B_{pq}\partial N^{pq}\delta(BN)) \tag{4.31}$$

$$= T(u)\Lambda_B^{mn}(u) + Q\Omega_B^{mn}(u) + T(u)\int_u^z dv\partial\Lambda_B^{mn}(v)$$

$$= T(u)\Lambda_B^{mn}(z) + Q\Omega_B^{mn}(u).$$

Since $T(u)\Lambda_B^{mn}(z)$ produces a total derivative with respect to the Teichmuller parameter τ_P associated to the Beltrami differential $\mu_P(u)$, the Lorentz variation of b_B only changes the scattering amplitude by a surface term.

5. Multiloop Amplitude Prescription

Using the picture-changing operators of section 3 and the b_B ghost of section 4, one can give a super-Poincaré covariant prescription for computing N-point g-loop closed superstring scattering amplitudes as

$$\mathcal{A} = \int d^2\tau_1 ... d^2\tau_{3g-3} \langle \mid \prod_{P=1}^{3g-3} \int d^2 u_P \mu_P(u_P) \widetilde{b}_{B_P}(u_P, z_P) \tag{5.1}$$

$$\prod_{P=3g-2}^{10g} Z_{B_P}(z_P) \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \mid^2 \prod_{T=1}^{N} \int d^2 t_T U_T(t_T) \rangle,$$

where $|\ \ |^2$ signifies the left-right product, τ_P are the Teichmuller parameters associated to the Beltrami differentials $\mu_P(u_P)$, and $U_T(t_T)$ are the dimension $(1,1)$ closed string vertex operators for the N external states. The constant antisymmetric tensors B_P^{mn} in b_{B_P} and Z_{B_P} will be chosen to satisfy (3.17) and will also be chosen such that $B_I = B_{I+10} = ... = B_{I+10(g-1)}$ for $I=1$ to 10. In other words, there will be ten constant antisymmetric tensors B_I^{mn}, each of which appear in g picture-raising operators or b_B ghosts. One possible choice for these ten tensors is $B_{I\ mn}N^{mn} = \delta_{I\ [ab]}N^{[ab]}$ where $a,b = 1$ to 5, $[ab]$ is a ten-component representation of $SU(5)$, and $N^{[ab]}$ transforms as a 10 representation under the SU(5) subgroup of the (Wick-rotated) Lorentz group SO(10).[21]

[21] In terms of the U(5)-covariant variables of [14], this choice would imply $B_{I\ mn}N^{mn} = \delta_{I\ [ab]}v^{[ab]}$.

When $g = 1$, the prescription of (5.1) needs to be modified for the usual reason that genus-one worldsheets are invariant under constant translations, so one of the vertex operators should be unintegrated. The one-loop amplitude prescription is therefore

$$\mathcal{A} = \int d^2\tau \langle \mid \int d^2u \mu(u) \widetilde{b}_{B_1}(u, z_1) \tag{5.2}$$

$$\prod_{P=2}^{10} Z_{B_P}(z_P) Z_J(v) \prod_{I=1}^{11} Y_{C_I}(y_I) \mid^2 V_1(t_1) \prod_{T=2}^{N} \int d^2 t_T U_T(t_T) \rangle,$$

where $V_1(t_1)$ is the unintegrated closed string vertex operator.

As shown in the previous sections, the Lorentz variations of $\widetilde{b}_{B_P}$, Z_{B_P} and Y_{C_I} are BRST-trivial, so the prescriptions of (5.1) and (5.2) are Lorentz-invariant up to possible surface terms. Also, all operators in (5.1) and (5.2) are manifestly spacetime supersymmetric except for Y_{C_I}, whose supersymmetry variation is BRST-trivial. In section 6, it will be argued that surface terms can be ignored in this formalism because of finiteness properties of the correlation functions. So the amplitude prescriptions of (5.1) and (5.2) are super-Poincaré covariant. This implies that $\mathcal{A}$ is independent of the eleven constant spinors C_I and ten constant tensors B_P which appear in the picture-changing operators. As will now be shown, functional integration over the matter fields and pure spinor ghosts can be used to derive manifestly Lorentz-covariant expressions from the amplitude prescriptions of (5.1) and (5.2).

As usual, the functional integration factorizes into partition functions and correlation functions for the different worldsheet variables. However, in the pure spinor formalism, the partition functions for the different worldsheet variables cancel each other out. This is easy to verify since the partition function for the ten bosonic x^μ variables gives a factor of $(\det \overline{\partial}_0)^{-5} (\det \partial_0)^{-5}$ where ∂_0 and $\overline{\partial}_0$ are the holomorphic and antiholomorphic derivatives acting on fields of zero conformal weight, the partition function for the sixteen fermionic $(\theta^\alpha, p_\alpha)$ and $(\overline{\theta}^\alpha, \overline{p}_\alpha)$ variables gives a factor of $(\det \overline{\partial}_0)^{16} (\det \partial_0)^{16}$, and the partition function for the eleven bosonic $(\lambda^\alpha, w_\alpha)$ and $(\overline{\lambda}^\alpha, \overline{w}_\alpha)$ variables gives a factor of $(\det \overline{\partial}_0)^{-11} (\det \partial_0)^{-11}$. So to perform the functional integral, one only needs to compute the correlation functions for the matter variables and pure spinor ghosts.

5.1. Correlation function for matter variables

In computing the g-loop correlation functions, one can follow the same general procedure as in the tree amplitude computation of subsection (3.3), but one now needs to take into account the g zero modes of the fields with $+1$ conformal weight. For example, one can functionally integrate over the $(\theta^\alpha, p_\alpha)$ variables by first separating off the zero mode of d_α by writing

$$d_\alpha(z) = \sum_{R=1}^{g} d_\alpha^R \omega_R(z) + \widehat{d}_\alpha(z) \tag{5.3}$$

where ω_R are the g holomorphic one-forms and d_α^R are the $16g$ zero modes of d_α. Since the poles of $\widehat{d}_\alpha(z)$ are determined by its OPE's with the other fields, these OPE's completely fix the dependence of the correlation function on the location z.

For example, the g-loop analog of the correlation function of (2.21) is given by

$$\langle d_\alpha(z)\Pi^m(u)\partial\theta^\beta(v)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle \tag{5.4}$$

$$= d_\alpha^R\omega_R(z)\langle\Pi^m(u)\partial\theta^\beta(v)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle$$

$$+\gamma^m_{\alpha\rho}F(z,u)\langle\partial\theta^\rho(u)\partial\theta^\beta(v)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle$$

$$+\delta_\alpha^\beta\partial_v F(z,v)\langle\Pi^m(u)d_\gamma(w)A_\delta(x(y),\theta(y))\rangle$$

$$+\gamma^n_{\alpha\gamma}F(z,w)\langle\Pi^m(u)\partial\theta^\beta(v)\Pi^n(w)A_\delta(x(y),\theta(y))\rangle$$

$$+F(z,y)\langle\Pi^m(u)\partial\theta^\beta(v)\delta_\gamma(w)D_\alpha A_\delta(x(y),\theta(y))\rangle$$

where

$$F(z,y) = \partial_z \log E(z,y)$$

and $E(z,y)$ is the holomorphic prime form which goes like $(z-y)$ when z approaches y [43][44].

Using this procedure, one can remove the d_α's one at a time from the correlation function. After all the d_α's have been removed, one can replace all remaining θ's in the correlation function by their zero mode. One then functionally integrates over the 16 θ^α zero modes and the $16g$ d_α^R zero modes using standard Berezin integration.

Although the functions $F(z,y)$ coming from the $\widehat{d}_\alpha$ OPE's in (5.4) are not single-valued when either z or y goes around a B-cycle of the genus g surface, the scattering amplitude will be single-valued. This is because when $d_\alpha(z)$ goes around the R^{th} B-cycle,

$F(z,y) \to F(z,y) - 2\pi i \omega_R(z)$, which induces a change in the correlation function of (5.4) by the term

$$-2\pi i \omega_R(z) \langle \oint ds d_\alpha(s)\ \Pi^m(u) \partial\theta^\beta(v) d_\gamma(w) A_\delta(x(y),\theta(y)) \rangle \tag{5.5}$$

where the contour integral of s goes around all the other points on the surface. Since d_α is a conserved current, the contour integral can be deformed off the back of the surface, giving no contribution to the scattering amplitude.

And when $\theta^\alpha(y)$ goes around the R^{th} B-cycle, $F(z,y) \to F(z,y) + 2\pi i \omega_R(z)$. Since this change in $F(z,y)$ is independent of y and is proportional to $\omega_R(z)$, the resulting change in the d_α correlation function can be cancelled by shifting the d_α^R zero mode in (5.3) by a z-independent amount. Since Berezin integration is unchanged by a constant shift of the Grassmann variables, the scattering amplitude is single-valued after integration over the d_α^R zero mode.

After performing functional integration over the $(\theta^\alpha, p_\alpha)$ variables in this manner, one can easily perform functional integration over the x^m variables using the standard techniques [43][44]. For example, the correlation function $\langle \prod_{r=1}^N \exp(ik\cdot x(u_r)) \rangle$ is equal to

$$= \prod_{R=1}^{g} \int d^{10}P_R \mid \exp(i\pi P_R \cdot P_S \tau_{RS} + 2\pi i \sum_{r=1}^{N} k_r \cdot P_R \int^{u_r} dv \omega_R(v)) \prod_{r<s} E(u_r,u_s)^{k_r\cdot k_s} \mid^2 \tag{5.6}$$

where P_R^m is the loop momentum through the R^{th} A-cycle, τ_{RS} is the period matrix, and $E(u,v)$ is the holomorphic prime form.

5.2. Correlation function for the pure spinor ghosts

After functionally integrating over the matter variables, one is left with a correlation function depending on the pure spinor ghost operators λ^α, N^{mn} and J. To compute this correlation function, first separate off the g zero modes of N_{mn} by writing

$$N_{mn}(z) = N_{mn}^R \omega_R(z) + \widehat{N}_{mn}(z). \tag{5.7}$$

Since the singularities of $\widehat{N}_{mn}(z)$ are determined from the OPE's of (2.6), the dependence of the correlation function on z is completely determined.

For example, a g-loop analog of the computation of (2.22) is

$$\langle N_{mn}(z) N_{pq}(u) \lambda^\alpha(v) \delta(BN(w)) \delta(C\lambda(y)) \rangle = \tag{5.8}$$

$$= N^R_{mn}\omega_R(z)\langle N_{pq}(u)\lambda^\alpha(v)\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$+F(z,u)\langle(\eta_{p[n}N_{m]q}(u)-\eta_{q[n}N_{m]p}(u))\lambda^\alpha(v)\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$-3\partial_u F(z,u)\langle\eta_{q[m}\eta_{n]p}\lambda^\alpha(v)\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$+\frac{1}{2}F(z,v)\langle N_{pq}(u)(\gamma_{mn}\lambda(v))^\alpha\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$+2F(z,w)\langle N_{pq}(u)\lambda^\alpha(v)B_{r[m}N_{n]}{}^r(w)\partial\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$+6\partial_w F(z,w)\langle N_{pq}(u)\lambda^\alpha(v)B_{mn}\partial\delta(BN(w))\delta(C\lambda(y))\rangle$$

$$+\frac{1}{2}F(z,y)\langle N_{pq}(u)\lambda^\alpha(v)\delta(BN(w))(C\gamma_{mn}\lambda(y))\partial\delta(C\lambda(y))\rangle.$$

If one counts $\partial^L\delta(BN)$ as containing $(-L)$ N's, then the number of N's is decreased after performing this correlation function. So repeating this procedure enough times will eventually give a correlation function with a net zero number of N's, at which point one can stop. Note that the procedure of separating off the zero mode of $N_{mn}(z)$ must also be used for the N_{mn} appearing in $\delta(BN)$. So one needs to include the contribution from

$$\delta(BN(z)) = \delta(BN^R\omega_R(z) + B\widehat{N}(z)) \tag{5.9}$$

$$= \delta(BN^R\omega_R(z)) + (B\widehat{N}(z))\partial\delta(BN^R\omega_R(z)) + \frac{1}{2}(B\widehat{N}(z))^2\partial^2\delta(BN^R\omega_R(z)) + ...,$$

where one uses the OPE of $\widehat{N}(z)$ with the other fields to determine the dependence of the correlation function on z.

As in the $(\theta^\alpha, p_\alpha)$ correlation function, although $F(z,y)$ is not single-valued when either z or y goes around a B-cycle, the scattering amplitude will be single-valued. When $N_{mn}(z)$ goes around the R^{th} B-cycle, the change in the correlation function of (5.8) is equal to

$$-2\pi i\omega_R(z)\langle(\oint ds N_{mn}(s))N_{pq}(u)\lambda^\alpha(v)\delta(BN(w))\delta(C\lambda(y))\rangle \tag{5.10}$$

where the contour integral of s goes around all points on the surface. Since N_{mn} is a conserved current, the contour can be deformed off the surface, so this contribution vanishes. And when $\lambda^\alpha(y)$ goes around a B-cycle, the change in the correlation function is independent of y and can be cancelled by an appropriate shift of the N^R_{mn} zero modes. So after integrating over the N^R_{mn} zero modes using a shift-invariant measure, this contribution will also vanish.

After removing all the N_{mn}'s from the correlation function and replacing them with N^R_{mn} zero modes, one can follow the same procedure for the $J(z)$'s in the correlation function. For example, after separating off the g zero modes by writing

$$J(z) = J^R \omega_R(z) + \widehat{J}(z),$$

one can use the OPE's of (2.6) to show that

$$\langle J(z)\delta(J(u))\lambda^{\alpha_1}(v_1)...\lambda^{\alpha_M}(v_M)\delta(C_1\lambda(y_1))...\delta(C_{11}\lambda(y_{11}))\rangle \tag{5.11}$$

$$= J^R\omega_R(z)\langle\delta(J(u))\lambda^{\alpha_1}(v_1)...\lambda^{\alpha_M}(v_M)\delta(C_1\lambda(y_1))...\delta(C_{11}\lambda(y_{11}))\rangle$$

$$-4\partial_u F(z,u)\langle\partial\delta(J(u))\lambda^{\alpha_1}(v_1)...\lambda^{\alpha_M}(v_M)\delta(C_1\lambda(y_1))...\delta(C_{11}\lambda(y_{11}))\rangle$$

$$+(\sum_{Q=1}^{M} F(z,v_Q) - \sum_{I=1}^{11} F(z,y_I) + 8\partial_z(\ln\sigma(z)))$$

$$\langle\delta(J(u))\lambda^{\alpha_1}(v_1)...\lambda^{\alpha_M}(v_M)\delta(C_1\lambda(y_1))...\delta(C_{11}\lambda(y_{11}))\rangle$$

where the term proportional to $8\partial_z(\ln\sigma(z))$ comes from OPE's with the screening charge which is responsible for the ghost-number anomaly. As discussed in [44], $\sigma(z)$ is a multi-valued holomorphic function without zeros or poles which satisfies

$$\partial_z \ln\sigma(z) \to \partial_z \ln\sigma(z) + 2\pi i(g-1)\omega_R(z) \tag{5.12}$$

when z goes around the R^{th} B-cycle. A convenient representation for $\partial_z(\ln\sigma(z))$ is [3]

$$\partial_z(\ln\sigma(z)) = \sum_{R=1}^{g}\oint_{A_R} dv\omega_R(v)F(z,v) \tag{5.13}$$

where $F(z,v) = \partial_z \ln E(z,v)$ and $E(z,v)$ is the holomorphic prime form.

One can easily use (5.12) and the ghost-number anomaly to show that (5.11) is invariant when $J(z)$ goes around the R^{th} B cycle.[22] And when $\lambda^\alpha(v)$ goes around the R^{th} B-cycle, the change in the correlation function can be cancelled by shifting the zero mode of J^R.

After removing all N's and J's from the correlation function and replacing them with N^R_{mn} and J^R zero modes, one can also replace all remaining λ^α's in the correlation function by their zero mode. As will now be described, one then needs to integrate over the $(\lambda^\alpha, N^R_{mn}, J^R)$ zero modes using the measure factors defined in subsection (3.1).

[22] Because of the ghost-number anomaly, J is not a conserved current and deforming $\oint dsJ(s)$ off the surface gives a contribution which is cancelled by (5.12).

5.3. Integral over pure spinor zero modes

After integrating out the non-zero modes of $(\lambda^\alpha, N_{mn}, J)$, one obtains an expression $\langle f(\lambda, N_R, J_R, C_I, B_P)\rangle$ depending only on the zero modes of $(\lambda^\alpha, N^R_{mn}, J^R)$ and the constant spinors and tensors C_I and B_P^{mn}. The scattering amplitude is then defined by the integral

$$\mathcal{A} = \int [\mathcal{D}\lambda][\mathcal{D}N_1]...[\mathcal{D}N_g] f(\lambda^\alpha, N^R_{mn}, J^R, C^I, B^P_{mn}) \tag{5.14}$$

where $[\mathcal{D}\lambda]$ and $[\mathcal{D}N]$ are defined in (3.5) and (3.9).

Using the properties of the measure factors $[\mathcal{D}\lambda]$ and $[\mathcal{D}N]$, one can write

$$\mathcal{A} = \int [\mathcal{D}\lambda] \prod_{R=1}^{g} [\mathcal{D}N_R] f = \int (d^{11}\lambda)^{[\alpha_1...\alpha_{11}]} \prod_{R=1}^{g} (d^{11}N_R)^{[[m_1^R n_1^R]...[m_{10}^R n_{10}^R]]} \tag{5.15}$$

$$\prod_{R=1}^{g} (\gamma_{m_1^R n_1^R m_2^R m_3^R m_4^R})^{\rho_1^R \rho_2^R} (\gamma_{m_5^R n_5^R n_2^R m_6^R m_7^R})^{\rho_3^R \rho_4^R} (\gamma_{m_8^R n_8^R n_3^R n_6^R m_9^R})^{\rho_5^R \rho_6^R} (\gamma_{m_{10}^R n_{10}^R n_4^R n_7^R n_9^R})^{\rho_7^R \rho_8^R}$$

$$(\epsilon \mathcal{T}^{-1})^{((\beta_1\beta_2\beta_3))}_{[\alpha_1...\alpha_{11}]} f_{((\beta_1\beta_2\beta_3\rho_1^1...\rho_8^1...\rho_1^g...\rho_8^g))}(\lambda, N_R, J_R, C_I, B_P)$$

where

$$f = \lambda^{\beta_1}\lambda^{\beta_2}\lambda^{\beta_3}\lambda^{\rho_1^1}...\lambda^{\rho_8^1}...\lambda^{\rho_1^g}...\lambda^{\rho_8^g} f_{((\beta_1\beta_2\beta_3\rho_1^1...\rho_8^1...\rho_1^g...\rho_8^g))}(\lambda, N_R, J_R, C_I, B_P).$$

As in the discussion of subsection (3.3) for tree amplitudes, (5.15) is in general a complicated function of the B_P's and C_I's. However, using the properties of the picture-changing operators and b_B ghost, one knows that the scattering amplitude must be independent of these constant spinors and tensors. One can therefore integrate (5.15) over all choices of B_P and C_I using a measure factor $[\mathcal{D}B][\mathcal{D}C]$ which satisfies $\int [\mathcal{D}B][\mathcal{D}C] = 1$. Note that (5.15) is manifestly invariant under rescalings of $C_{I\alpha}$ and B_P^{mn}, so these constant spinors and tensors can be interpreted as projective variables.

Using arguments similar to those of subsection (3.3), a manifestly Lorentz-covariant prescription will now be given for evaluating $\langle f(\lambda, N_R, J_R, C_I, B_P)\rangle$. To be non-vanishing and have ghost-number $8g-8$, $f(\lambda, N_R, J_R, C_I, B_P)$ will depend on $(\lambda, N_R, J_R, C_I, B_P)$ as

$$f(\lambda, N_R, J_R, C_I, B_P) = \tag{5.16}$$

$$h(\lambda, N_R, J_R, C_I, B_P) \prod_{R=1}^{g} \partial^{M_R}\delta(J_R) \prod_{P=1}^{10}\prod_{R=1}^{g} \partial^{L_{P,R}}\delta(B_P N_R)) \prod_{I=1}^{11} \partial^{K_I}(C_I\lambda),$$

where h is a polynomial depending on $(\lambda^\alpha, N^R_{mn}, J^R, C_{I\alpha}, B^{mn}_P)$ as

$$(\lambda)^{8g-8+\sum_{I=1}^{11}(K_I+1)} \prod_{R=1}^{g} (J_R)^{M_R} (N_R)^{\sum_{P=1}^{10} L_{P,R}} \prod_{P=1}^{10} (B_P)^{\sum_{R=1}^{g}(L_{P,R}+1)} \prod_{I=1}^{11} (C_I)^{K_I+1}. \tag{5.17}$$

Using Lorentz invariance and symmetry properties, one can argue that

$$\mathcal{A} = \int [\mathcal{D}B][\mathcal{D}C] \int [\mathcal{D}\lambda] \prod_{R=1}^{g} [\mathcal{D}N_R] f(\lambda, N_R, J_R, C_I, B_P) \tag{5.18}$$

$$= c' \; [\; (\frac{\partial}{\partial\lambda}\gamma^{m_1 n_1 m_2 m_3 m_4}\frac{\partial}{\partial\lambda})(\frac{\partial}{\partial\lambda}\gamma^{m_5 n_5 n_2 m_6 m_7}\frac{\partial}{\partial\lambda})(\frac{\partial}{\partial\lambda}\gamma^{m_8 n_8 n_3 n_6 m_9}\frac{\partial}{\partial\lambda})$$

$$(\frac{\partial}{\partial\lambda}\gamma^{m_{10} n_{10} n_4 n_7 n_9}\frac{\partial}{\partial\lambda}) \; \frac{\partial}{\partial B_1^{m_1 n_1}} \cdots \frac{\partial}{\partial B_{10}^{m_{10} n_{10}}} \;]^g$$

$$(\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \ldots \rho_{11}]} \frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\lambda^\beta}\frac{\partial}{\partial\lambda^\gamma}\frac{\partial}{\partial C_{1\rho_1}} \cdots \frac{\partial}{\partial C_{11\rho_{11}}}$$

$$\prod_{I=1}^{11} (\frac{\partial}{\partial\lambda^\delta}\frac{\partial}{\partial C_{I\delta}})^{K_I} \prod_{P=1}^{10}\prod_{R=1}^{g} (\frac{\partial}{\partial B_P^{pq}}\frac{\partial}{\partial N_{Rpq}})^{L_{P,R}} \prod_{R=1}^{g} (\frac{\partial}{\partial J_R})^{M_R} h(\lambda, N_R, J_R, C_I, B_P),$$

where the proportionality constant c' can be computed as in (3.26).

So as claimed, the final expression for the scattering amplitude is a manifestly Lorentz-covariant function of the polarizations and momenta of the external states. Although this expression is complicated for arbitrary g-loop amplitudes, it will be shown in the following section how this prescription can be used to prove certain vanishing theorems.

6. Amplitude Computations and Vanishing Theorems

In this section, the amplitude prescription of section 5 will be used to prove certain properties of closed superstring scattering amplitudes involving massless states. In subsection (6.1), the closed superstring vertex operator for Type IIB supergravity states will be reviewed. In subsection (6.2), it will be proven that massless N-point g-loop amplitudes are vanishing whenever $N < 4$ and $g > 0$. In subsection (6.3), the four-point massless one-loop amplitude will be computed. And in subsection (6.4), it will be proven that the low-energy limit of the four-point massless amplitude gets no perturbative contributions above one-loop.

6.1. Type IIB supergravity vertex operator

Just as the super-Maxwell states of the open superstring are described by the unintegrated vertex operator $V = \lambda^\alpha A_\alpha(x,\theta)$ satisfying $QV = 0$ and $\delta V = Q\Omega$, the Type IIB supergravity states of the closed superstring are described by the unintegrated vertex operator

$$V = \lambda^\alpha \overline{\lambda}^\beta A_{\alpha\beta}(x,\theta,\overline{\theta}) \tag{6.1}$$

satisfying

$$QV = \overline{Q}V = 0, \quad \delta V = Q\Omega + \overline{Q\Omega} \tag{6.2}$$

where $\overline{Q}\Omega = Q\overline{\Omega} = 0$. The equations $QV = \overline{Q}V = 0$ imply that

$$D_\gamma A_{\alpha\beta} + D_\alpha A_{\gamma\beta} = \gamma^m_{\alpha\gamma} A_{m\beta}, \quad \overline{D}_\gamma A_{\alpha\beta} + \overline{D}_\beta A_{\alpha\gamma} = \gamma^m_{\beta\gamma} A_{\alpha m} \tag{6.3}$$

for some superfields $A_{m\beta}$ and $A_{\alpha m}$ where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha \partial_m, \quad \overline{D}_\alpha = \frac{\partial}{\partial\overline{\theta}^\alpha} + \frac{1}{2}(\gamma^m\overline{\theta})_\alpha \partial_m \tag{6.4}$$

are the N=2 D=10 superspace derivatives. And the gauge transformations $\delta V = Q\Omega + \overline{Q\Omega}$ where $\overline{Q}\Omega = Q\overline{\Omega} = 0$ implies

$$\delta A_{\alpha\beta} = D_\alpha \Omega_\beta + \overline{D}_\beta \overline{\Omega}_\alpha, \tag{6.5}$$

$$\delta A_{m\alpha} = \partial_m \Omega_\alpha - \overline{D}_\alpha \overline{\Omega}_m, \quad \delta A_{\alpha m} = \partial_m \overline{\Omega}_\alpha - D_\alpha \Omega_m,$$

where

$$D_{(\alpha}\overline{\Omega}_{\beta)} = \gamma^m_{\alpha\beta}\overline{\Omega}_m, \quad \overline{D}_{(\alpha}\Omega_{\beta)} = \gamma^m_{\alpha\beta}\Omega_m.$$

In components, one can use (6.3) and (6.5) to gauge $A_{\alpha\beta}(x,\theta,\overline{\theta})$ to the form

$$A_{\alpha\beta}(x,\theta,\overline{\theta}) = e^{ik\cdot x}[h_{mn}(\gamma^m\theta)_\alpha(\gamma^n\overline{\theta})_\beta + \overline{\psi}^\gamma_m(\gamma^m\theta)_\alpha(\gamma^n\overline{\theta})_\beta(\gamma_n\overline{\theta})_\gamma \tag{6.6}$$

$$+\psi^\gamma_n(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^n\overline{\theta})_\beta + F^{\gamma\delta}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^n\overline{\theta})_\beta(\gamma_n\overline{\theta})_\delta + ...]$$

where

$$k^2 = k^m h_{mn} = k^n h_{mn} = k^m \overline{\psi}^\alpha_m = k^n(\gamma_n\overline{\psi}_m)_\alpha = 0,$$

$$k^m \psi^\alpha_m = k^n(\gamma_n\psi_m)_\alpha = k_m\gamma^m_{\alpha\gamma}F^{\gamma\delta} = k_m\gamma^m_{\alpha\delta}F^{\gamma\delta} = 0,$$

and ... involves products of k^m with h^{mn}, ψ_m^α, $\overline{\psi}_m^\alpha$, or $F^{\alpha\beta}$. So $A_{\alpha\beta}(x,\theta,\overline{\theta})$ describes the on-shell Type IIB supergravity multiplet where h_{mn} describes the graviton, antisymmetric tensor and dilaton, ψ_m^α and $\overline{\psi}_m^\alpha$ describe the gravitini and dilatini, and $F^{\alpha\beta}$ describe the Ramond-Ramond field strengths. Note that the x-independent part of (6.6) can be interpreted as the left-right product of two super-Maxwell superfields $A_\alpha(\theta)\overline{A}_\beta(\overline{\theta})$ where $A_\alpha(x,\theta) = a_m(\gamma^m\theta) + \xi^\gamma(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma + ...$ and

$$h_{mn} = a_m\overline{a}_n, \quad \overline{\psi}_m^\alpha = a_m\overline{\xi}^\alpha, \quad \psi_m^\alpha = \xi^\alpha\overline{a}_m, \quad F^{\alpha\beta} = \xi^\alpha\overline{\xi}^\beta.$$

So one can interpret the unintegrated massless closed superstring vertex operator of (6.1) as the left-right product of two unintegrated massless open superstring vertex operators using the identification

$$\lambda^\alpha\overline{\lambda}^\beta A_{\alpha\beta}(x,\theta,\overline{\theta}) = e^{ik\cdot x}\lambda^\alpha A_\alpha(\theta)\overline{\lambda}^\beta\overline{A}_\beta(\overline{\theta}).$$

Just as the integrated open superstring vertex operator U_{open} is related to the unintegrated open superstring vertex operator V_{open} by $QU_{open} = \partial V_{open}$, the integrated closed superstring vertex operator U_{closed} is related to the unintegrated closed superstring vertex operator V_{closed} by $Q\overline{Q}U_{closed} = \partial\overline{\partial}V_{closed}$. Although one can easily write an explicit expression for the integrated form of the Type IIB supergravity vertex operator [19][45], it will be more convenient to recognize that it is related to the left-right product of two integrated super-Maxwell vertex operators of (2.19). So the integrated Type IIB supergravity vertex operator can be expressed as

$$U_{closed} = e^{ik\cdot x}(\partial\theta^\alpha A_\alpha(\theta) + \Pi^m A_m(\theta) + d_\alpha W^\alpha(\theta) + \frac{1}{2}N^{mn}\mathcal{F}_{mn}(\theta)) \tag{6.7}$$

$$(\overline{\partial\theta}^\beta\overline{A}_\beta(\overline{\theta}) + \overline{\Pi}^p\overline{A}_p(\overline{\theta}) + \overline{d}_\beta\overline{W}^\beta(\overline{\theta}) + \frac{1}{2}\overline{N}^{pq}\overline{\mathcal{F}}_{pq}(\overline{\theta})).$$

Since the closed string graviton h_{mn} is identified with the product of $a_m\overline{a}_n$, the $\theta = \overline{\theta} = 0$ component of $\mathcal{F}_{mn}(\theta)\overline{\mathcal{F}}_{pq}(\overline{\theta})$ is identified with the linearized curvature tensor $R_{mnpq} = k_{[m}h_{n][q}k_{p]}$.

6.2. Non-renormalization theorem

In this subsection, the amplitude prescription of section 5 will be used to prove that massless N-point g-loop amplitudes vanish whenever $N < 4$ and $g > 0$. For $N = 0$, this implies vanishing of the cosmological constant; for $N = 1$, it implies absence of tadpoles; for $N = 2$, it implies the mass is not renormalized; and for $N = 3$, it implies the coupling constant is not renormalized. Using the arguments of [4][32] which were summarized in the introduction, and assuming factorization and the absence of unphysical divergences in the interior of moduli space, these non-renormalization theorems imply that superstring scattering amplitudes are finite order-by-order in perturbation theory.

Although surface terms were ignored in deriving the amplitude prescription of section 5, it is necessary that the proof of the non-renormalization theorem remain valid even if one includes such surface term contributions. Otherwise, there could be divergent surface term contributions which would invalidate the proof. For this reason, one cannot assume Lorentz invariance or spacetime supersymmetry to prove the non-renormalization theorem since the prescription of (5.1) is Lorentz invariant and spacetime supersymmetric only after ignoring the surface terms.

Fortunately, it will be possible to prove the non-renormalization theorem using only the counting of zero modes. Since this type of argument implies the pointwise vanishing of the integrand of the scattering amplitude (as opposed to only implying that the integrated amplitude vanishes), the proof remains valid if one includes the contribution of surface terms.

On a surface of arbitrary genus, one needs 16 zero modes of θ^α and $\overline{\theta}^\alpha$ for the amplitude to be non-vanishing. Since the only operators in (5.1) containing θ^α zero modes[23] are the eleven Y_C picture-lowering operators and the U_T vertex operators, and since each Y_C contributes a single θ^α zero mode, the U_T vertex operators must contribute at least five θ^α and five $\overline{\theta}^\alpha$ zero modes for the amplitude to be non-vanishing. This immediately implies that zero-point amplitudes vanish.

[23] When expressed in terms of the free fields $(x^m, \theta^\alpha, p_\alpha)$, Π^m and d_α contain θ's without derivatives which naively could contribute θ^α zero modes. But if the supersymmetric OPE's of (2.11) are used to integrate out the non-zero worldsheet modes, the OPE's involving Π^m and d_α will never produce θ^α zero modes.

For one-point amplitudes, conservation of momentum implies that the external state must have momentum $k^m = 0$. But when $k^m = 0$, the maximum number of zero modes in the vertex operator is one θ^α and one $\overline{\theta}^\alpha$ coming from the superfield

$$A_{\alpha\beta}(\theta,\overline{\theta}) = h_{mn}(\gamma^m\theta)_\alpha(\gamma^n\overline{\theta})_\beta.$$

All other components in the superfields appearing in the vertex operators of (6.1) and (6.7) are either fermionic or involve powers of k^m. So all one-point amplitudes vanish.

To prove that massless two and three-point amplitudes vanish for non-zero g, one needs to count the available zero modes of d_α, as well as the zero modes of N_{mn}. On a genus g surface, non-vanishing amplitudes require $16g$ zero modes of d_α. In addition, the number of N_{mn} zero modes must be at least as large as the number of derivatives acting on the delta functions $\delta(BN)$ in the amplitude prescription. Otherwise, integration over the N^{mn} zero modes will trivially vanish.

To prove the N-point g-loop non-renormalization theorem for $N = 2$ and $N = 3$, it is useful to distinguish between one-loop amplitudes and multiloop amplitudes. For massless N-point one-loop amplitudes using the prescription of (5.2), there are $(N-1)$ integrated vertex operators of (6.7), each of which can either provide a d_α zero mode or an N_{mn} zero mode. So one has at most $(N-1-M)$ d_α zero modes and M N_{mn} zero modes coming from the vertex operators where $M \leq N-1$. Each of the nine Z_{B_P} operators and one Z_J operator can provide a single d_α zero mode, so to get a total of 16 d_α zero modes, b_B must provide at least

$$16 - (N-1-M) - 9 - 1 = 7 - N + M \tag{6.8}$$

d_α zero modes.

It is easy to verify from (4.29) that b_B can provide a maximum of four d_α zero modes, however, the terms containing four d_α zero modes also contain (-1) N_{mn} zero modes where a derivative acting on $\delta(BN)$ counts as a negative N_{mn} zero mode. This fact can easily be derived from the $+4$ engineering dimension of b_B where $[\lambda^\alpha, \theta^\alpha, x^m, d_\alpha, N_{mn}]$ are defined to carry engineering dimension $[0, \frac{1}{2}, 1, \frac{3}{2}, 2]$ and $\partial^L\delta(BN)$ is defined to carry engineering dimension $-2L$. Since $(d)^4$ carries engineering dimension $+6$, it can only appear in b_B together with a term such as $\partial\delta(BN)$ which carries engineering dimension -2.

So for $N \leq 3$ and $M = 0$, (6.8) implies that the only way to obtain 16 d_α zero modes is if b_B provides at least four d_α zero modes. But in this case, b_B contains (-1) N_{mn} zero modes, so the amplitudes vanish since there are not enough N_{mn} zero modes to absorb

the derivatives on $\delta(BN)$. And when $M > 0$, the amplitude vanishes for $N \leq 3$ since one needs more than four d_α zero modes to come from b_B.

For multiloop amplitudes, the argument is similar, but one now has N integrated vertex operators instead of $(N-1)$. So the vertex operators can contribute a maximum of $(N-M)$ d_α zero modes and M N_{mn} zero modes where $M \leq N$. And each of the $7g+3$ Z_B and g Z_J operators can provide a single d_α zero mode. So to get a total of $16g$ d_α zero modes, the $(3g-3)$ b_B's must provide at least

$$16g - (N-M) - (7g+3) - g = 8g - 3 - N + M \tag{6.9}$$

d_α zero modes. Since $(3g-3)$ b_B's carry engineering dimension $12g-12$, d_α carries engineering dimension $\frac{3}{2}$, and N_{mn} carries engineering dimension $+2$, the $(3g-3)$ b_B's can provide a maximum of $(8g-8)$ d_α zero modes with no derivatives of $\delta(BN)$, or $(8g-8+\frac{4}{3}M)$ d_α zero modes with M derivatives of $\delta(BN)$. Since

$$8g - 8 + \frac{4}{3}M < 8g - 3 - N + M \tag{6.10}$$

whenever $M \leq N \leq 3$, there is no way for the $(3g-3)$ b_B's to provide enough d_α zero modes without providing too many derivatives of $\delta(BN)$.

So the N-point multiloop non-renormalization theorem has been proven for $N \leq 3$. Note that when $N = 4$,

$$8g - 8 + \frac{4}{3}M \geq 8g - 3 - N + M \tag{6.11}$$

if one chooses $M = 3$ or $M = 4$. So four-point multiloop amplitudes do not need to vanish. However, as will be shown in subsection (6.4), one can prove that the low-energy limit of these multiloop amplitudes vanish, which implies that the R^4 term in the effective action gets no perturbative corrections above one loop. But before proving this, it will be useful to see how the four-point one-loop amplitude is reproduced in the pure spinor formalism.

6.3. Massless four-point one-loop amplitude

The simplest non-vanishing one-loop amplitude involves four massless particles and can be computed using either the RNS or light-cone GS formalism. Nevertheless, it is interesting to see how this well-known amplitude can be derived from the super-Poincaré covariant prescription of section 5.

As discussed in (6.8), b_B must provide at least $(7-N+M)$ d_α zero modes for the one-loop amplitude to be non-vanishing where N is the number of external states and M

is the number of N_{mn} zero modes coming from the vertex operators. Since b_B carries engineering dimension +4, the only way to satisfy (6.8) when $N = 4$ is if $M = 1$ and b_B provides four d_α zero modes. In terms of the operators $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$ defined in (4.10), (4.14) and (4.18), the only such terms in b_B are

$$\frac{1}{4}H^{\beta\alpha}(Bd)_\alpha(Bd)_\beta\partial\delta(BN) - \frac{1}{8}K^{\gamma\beta\alpha}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma\partial^2\delta(BN) \tag{6.12}$$

$$+\frac{1}{16}L^{\delta\gamma\beta\alpha}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma(Bd)_\delta\partial^3\delta(BN)$$

where $(Bd)_\alpha = B^{mn}(\gamma_{mn}d)_\alpha$.

Since all d_α and N_{mn} variables are used to absorb zero modes in this correlation function, the functional integral over the $(\theta^\alpha, p_\alpha)$ variables and over the pure spinor ghosts only contributes to the four-point one-loop amplitude through the zero-mode integral

$$|\int d^{16}\theta \int d^{16}d \int [\mathcal{D}\lambda][\mathcal{D}N] \tag{6.13}$$

$$(-\frac{1}{1536}\gamma^{\beta\alpha}_{mnp}(d\gamma^{mnp}d)(Bd)_\alpha(Bd)_\beta\partial\delta(BN) + \frac{1}{8}c^{\gamma\beta\alpha\rho}_{1mn}N^{mn}d_\rho(Bd)_\alpha(Bd)_\beta(Bd)_\gamma\partial^2\delta(BN)$$

$$-\frac{1}{16}c^{\delta\gamma\beta\alpha}_{4mnpq}N^{mn}N^{pq}(Bd)_\alpha(Bd)_\beta(Bd)_\gamma(Bd)_\delta\partial^3\delta(BN)\ )$$

$$\prod_{P=2}^{10} B_P^{mn}(\lambda\gamma_{mn}d)\delta(B_P N)(\lambda d)\delta(J)\prod_{I=1}^{11}(C_I\theta)\delta(C_I\lambda)$$

$$\lambda^\alpha A_{1\alpha}(\theta)\prod_{T=2}^{4}(d_\alpha W_T^\alpha(\theta) + \frac{1}{2}N_{mn}\mathcal{F}_T^{mn}(\theta))\ |^2$$

where the closed superstring vertex operators have been written as the left-right product of open superstring vertex operators as in (6.7).

Integrating over the constant spinors and tensors $C_{I\alpha}$ and B_P^{mn} and using the formula of (5.18), one finds that the term in (6.13) which is independent of $c^{\gamma\beta\alpha\rho}_{1mn}$ and $c^{\delta\gamma\beta\alpha}_{4mnpq}$ is proportional to

$$|\int d^{16}\theta \int d^{16}d\ (\epsilon\mathcal{T}^{-1})^{((\kappa_1\kappa_2\kappa_3}_{[\rho_1\ldots\rho_{11}]} \tag{6.14}$$

$$(\gamma^{m_1n_1m_2m_3m_4})^{\kappa_4\kappa_5}(\gamma^{m_5n_5n_2m_6m_7})^{\kappa_6\kappa_7}(\gamma^{m_8n_8n_3n_6m_9})^{\kappa_8\kappa_9}(\gamma^{m_{10}n_{10}n_4n_7n_9})^{\kappa_{10}\kappa_{11}))}$$

$$(d\gamma^{rst}d)\gamma^{\sigma\gamma}_{rst}(\gamma_{pq}d)_\sigma(\gamma_{m_1n_1}d)_\gamma(\gamma_{m_2n_2}d)_{\kappa_2}\ldots(\gamma_{m_{10}n_{10}}d)_{\kappa_{10}}d_{\kappa_{11}}(\theta^{\rho_1}\ldots\theta^{\rho_{11}})$$

$$A_{1\kappa_1}(\theta)\ (\ d_\alpha W_2^\alpha(\theta)d_\beta W_3^\beta(\theta)\mathcal{F}_4^{pq}(\theta)$$

$$+d_\alpha W_3^\alpha(\theta) d_\beta W_4^\beta(\theta)\mathcal{F}_2^{pq}(\theta) + d_\alpha W_4^\alpha(\theta) d_\beta W_2^\beta(\theta)\mathcal{F}_3^{pq}(\theta)\ )\ |^2$$

where the proportionality constant will not be determined here. Integrating over the d_α zero modes and performing gamma-matrix manipulations, one finds that this integral is proportional to

$$|\int d^{16}\theta(\epsilon\mathcal{T}^{-1})^{((\kappa_1\kappa_2\kappa_3))}_{[\rho_1...\rho_{11}]}(\theta^{\rho_1}...\theta^{\rho_{11}})(\gamma_{mnpqr})_{\kappa_1\kappa_2}A_{1\kappa_3}(\theta) \tag{6.15}$$

$$((W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta) + (W_3(\theta)\gamma^{mnp}W_4(\theta))\mathcal{F}_2^{qr}(\theta) + (W_4(\theta)\gamma^{mnp}W_2(\theta))\mathcal{F}_3^{qr}(\theta))|^2.$$

The result of (6.15) can be obtained without going through the complicated gamma-matrix manipulations by noting that the $\kappa_1\kappa_2\kappa_3$ indices on $(\epsilon\mathcal{T}^{-1})^{((\kappa_1\kappa_2\kappa_3))}_{[\rho_1...\rho_{11}]}(\theta^{\rho_1}...\theta^{\rho_{11}})$ need to be contracted with a γ-matrix traceless combination constructed from one A_α, two W^β's, and one F_{mn}. The only possible such combination is $(\gamma_{mnpqr})_{((\kappa_1\kappa_2}A_{\kappa_3))}(W\gamma^{pqr}W)F^{mn}$. For this reason, the terms in (6.13) which depend on $c_{1mn}^{\gamma\beta\alpha\rho}$ and $c_{4mnpq}^{\delta\gamma\beta\alpha}$ must also give contributions proportional to (6.15) after integration over $C_{I\alpha}$ and B_P^{mn}.

Finally, one needs to include the correlation function for the x^m variables as in (5.6) which gives the factor

$$\int d^{10}P|\exp(i\pi P^2\tau + 2\pi i\sum_{T=1}^{4}(k_T\cdot P)t_T)\prod_{T<U}E(t_T,t_U)^{k_T\cdot k_U}|^2 \tag{6.16}$$

$$= (Im\ \tau)^{-5}\prod_{T<U}G(t_T,t_U)^{k_T\cdot k_U}$$

where $G(t_T,t_U) = |E(t_T,t_U)|^2\exp(-2\pi(Im\ \tau)^{-1}(Im\ t_T)(Im\ t_U))$.

So up to a constant proportionality factor, the massless four-point one-loop amplitude is

$$\mathcal{A} = \int d^2\tau(Im\ \tau)^{-5}\int d^2t_2\int d^2t_3\int d^2t_4\prod_{T<U}G(t_T,t_U)^{k_T\cdot k_U} \tag{6.17}$$

$$|\int d^{16}\theta(\epsilon\mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]}\theta^{\rho_1}...\theta^{\rho_{11}}(\gamma_{mnpqr})_{\beta\gamma}A_{1\alpha}(\theta)$$

$$((W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta) + (W_3(\theta)\gamma^{mnp}W_4(\theta))\mathcal{F}_2^{qr}(\theta) + (W_4(\theta)\gamma^{mnp}W_2(\theta))\mathcal{F}_3^{qr}(\theta))|^2.$$

One can easily check that (6.17) is modular invariant and has a structure similar to the standard expression for the four-point one-loop amplitude. But because of the gauge superfield $A_{1\alpha}(\theta)$ in (6.17), $\mathcal{A}$ is not manifestly gauge invariant under

$$\delta A_{1\alpha}(\theta) = D_\alpha\Omega_1(\theta). \tag{6.18}$$

Nevertheless, one can use properties of pure spinors to show that the amplitude is in fact gauge invariant under (6.18). Integrating D_α by parts in the $\int d^{16}\theta$ integral, one obtains

$$\int d^{16}\theta(\epsilon\mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]}\delta A_{1\alpha}[\theta^{\rho_1}...\theta^{\rho_{11}}(\gamma_{mnpqr})_{\beta\gamma}(W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta)] \tag{6.19}$$

$$= -\int d^{16}\theta(\epsilon\mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]}\Omega_1(\theta)D_\alpha[\theta^{\rho_1}...\theta^{\rho_{11}}(\gamma_{mnpqr})_{\beta\gamma}(W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta)]$$

$$= \int d^{16}\theta(\epsilon\mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1...\rho_{11}]}\Omega_1(\theta)(\theta^{\rho_1}...\theta^{\rho_{11}})(\gamma_{mnpqr})_{\beta\gamma}D_\alpha[(W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta)]$$

where the identity $(\epsilon\mathcal{T}^{-1})^{((\rho_1\beta\gamma))}_{[\rho_1...\rho_{11}]} = 0$ was used. To compute

$$(\gamma_{mnpqr})_{((\beta\gamma}D_{\alpha))}[(W_2(\theta)\gamma^{mnp}W_3(\theta))\mathcal{F}_4^{qr}(\theta)], \tag{6.20}$$

note that Bianchi identities imply that $D_\alpha W^\delta = \frac{1}{4}(\gamma^{st})_\alpha{}^\delta\mathcal{F}_{st}$ and $D_\alpha\mathcal{F}^{qr} = \partial^{[q}(\gamma^{r]}W)_\alpha$. But $(\gamma_r)_{\alpha((\beta}\gamma^r_{\gamma))\delta} = 0$ implies that

$$(\gamma_{mnp}\gamma_{st})_{\delta((\alpha}(\gamma^{mnpqr})_{\beta\gamma))} = 0 \quad \text{and} \quad (\gamma_{mnpqr})_{((\beta\gamma}(\gamma^r)_{\alpha))\delta} = 0,$$

so (6.20) vanishes and $\mathcal{A}$ is gauge-invariant under (6.18).

Since the four external vertex operators need to provide 5 θ^α and 5 $\overline{\theta}^\alpha$ zero modes in (6.17), this amplitude implies the presence of a one-loop R^4 term in the low-energy effective action. To see this, note that using the left-right product language of (6.7), an R^4 term comes from $|F^4|^2$. In the $A_\alpha(\theta)$, $W^\alpha(\theta)$ and $\mathcal{F}_{mn}(\theta)$ superfields, the F_{mn} field strength is in the component

$$A_\alpha(\theta) = ... + (\theta\gamma^{mnp}\theta)(\gamma_p\theta)_\alpha F_{mn} + ..., \tag{6.21}$$

$$W^\alpha(\theta) = ... + (\gamma^{mn}\theta)^\alpha F_{mn} + ..., \quad \mathcal{F}_{mn}(\theta) = F_{mn} + ...$$

So if A_α provides three θ zero modes and each W^α provides one θ zero mode, one obtains an $|F^4|^2$ term from the vertex operators in (6.17). It should be straightforward to check that the contractions of the Lorentz indices in this $|F^4|^2$ term agrees with the usual contractions of the one-loop R^4 term in the effective action.

6.4. *Absence of multiloop R^4 contributions*

Although the four-point massless amplitude is expected to be non-vanishing at all loops, there is a conjecture based on S-duality of the Type IIB effective action that R^4 terms in the low-energy effective action do not get perturbative contributions above one-loop [29]. After much effort, this conjecture was recently verified in the RNS formalism at two loops [7][5]. As will now be shown, the multiloop prescription of section 5 can be easily used to prove the validity of this S-duality conjecture at all loops.

It was proven using (6.11) that the four-point massless multiloop amplitude vanishes unless at least three of the four integrated vertex operators contribute an N_{mn} zero mode. Since the only operators containing θ zero modes are the eleven picture-lowering operators and the external vertex operators, the functional integral over θ zero modes in the multiloop prescription for the four-point amplitude gives an expression of the form

$$|\int d^{16}\theta(\theta)^{11}(d_\alpha W_1^\alpha(\theta)+\frac{1}{2}N_{pq}\mathcal{F}_1^{pq}(\theta))\prod_{T=2}^{4}N_{mn}\mathcal{F}_T^{mn}(\theta)|^2. \tag{6.22}$$

Since the external vertex operators must contribute at least 5 θ^α and $\overline{\theta}^\alpha$ zero modes, and since F^{mn} appears in the component expansions of W^α and $\mathcal{F}^{mn}$ as in (6.21), one easily sees that there is no way to produce an $|F^4|^2$ term which would imply an R^4 term in the effective action. In fact, by examining the component expansion of the $\mathcal{F}_{mn}(\theta)$ and $W^\alpha(\theta)$ superfields, one finds that the term with fewest number of spacetime derivatives which contributes 5 θ's and 5 $\overline{\theta}$'s is $|(\partial F)(\partial F)F^2|^2$, which would imply a ∂^4R^4 contribution to the low-energy effective action.

So it has been proven that there are no multiloop contributions to R^4 terms (or ∂^2R^4 terms) in the low-energy effective action of the superstring. It should be noted that this proof has assumed that the correlation function over x^m does not contribute inverse powers of k^m which could cancel momentum factors coming from the θ integration in (6.22). Although the x^m correlation function does contain poles as a function of k^m when the external vertex operators collide, these poles only contribute to non-local terms in the effective action which involve massless propagators, and are not expected to contribute to local terms in the effective action such as the R^4 term.

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